Review Exam # 3, Math 4354
Chapter Sections: 12.3,12.6,12.8,13.1,14.3,14.4

Eigenvalue Problems: Find all eigenvalues λ_n and nonzero eigenfunctions $X_n(x)$ with homogeneous BCs.
$X'' + \lambda X = 0$, $X(0) = 0$, $X(L) = 0$, $X_n(x) = \sin(n\pi x/L)$, $\lambda_n = (n\pi/L)^2$, $n = 1, 2, \ldots$.
$X'' + \lambda X = 0$, $X'(0) = 0$, $X'(L) = 0$, $X_n(x) = \cos(n\pi x/L)$, $\lambda_n = (n\pi/L)^2$, $n = 0, 1, 2, \ldots$.
$X'' + \lambda X = 0$, $X(0) = 0$, $X'(L) = 0$, $X_n(x) = \sin((2n-1)\pi x/(2L))$, $\lambda_n = ((2n-1)\pi/(2L))^2$, $n = 1, 2, \ldots$.
$X'' + \lambda X = 0$, $X'(0) = 0$, $X(L) = 0$, $X_n(x) = \cos((2n-1)\pi x/(2L))$, $\lambda_n = ((2n-1)\pi/(2L))^2$, $n = 1, 2, \ldots$.

Heat Equation with lateral heat loss: $u_t = ku_{xx} - hu$, $0 < x < L$. Solution: $u(x,t) = e^{-ht}v(x,t)$, where v solves $v_t = kv_{xx}$, $0 < x < L$.

Nonhomogeneous PDEs and BCs:

$$\begin{aligned} &\text{PDE} \quad u_t = ku_{xx} + F(x),\ 0 < x < L,\ 0 < t < \infty \\ &\text{BC} \quad u(0,t) = u_0, \quad u(L,t) = u_1, \quad 0 < t < \infty \\ &\text{IC} \quad u(x,0) = f(x), \quad 0 < x < L. \end{aligned}$$

$$u(x,t) = v(x,t) + \psi(x)$$

Homogeneous $v(x,t)$	Nonhomogeneous (steady state) $\psi(x)$
PDE $v_t = kv_{xx}$, $0 < x < L$, $0 < t < \infty$	ODE $k\psi''(x) + F(x) = 0$, $0 < x < L$
BC $v(0,t) = 0$, $v(L,t) = 0$, $0 < t < \infty$	BC $\psi(0) = u_0$, $\psi(L) = u_1$

Apply IC to find constants $u(x,0) = v(x,0) + \psi(x) = f(x)$

Fourier Series in Two Variables. Solve the heat equation on a rectangular plate, $u(x,y,t)$ $u_t = k(u_{xx} + u_{yy})$ with homogeneous BCs on $x = 0, b$ and $y = 0, c$. Separation of variable leads to $T' + k(\lambda + \mu)T = 0$ and two eigenvalue problems, $X'' + \lambda X = 0$, $Y'' + \mu Y = 0$, with eigenvalues and eigenfunctions dependent on BCs. Double Fourier sine series or double Fourier cosine series.

$$u(x,y,t) = \sum_m \sum_n A_{mn} e^{-k(\lambda_m + \mu_n)t} X_m(x) Y_n(y)$$

Solve for A_{mn} from IC $u(x,y,0) = f(x,y)$.

Steady-State Temperature on a Circular Plate: $u(x,y) = u(r,\theta)$ with a radius of $r = c$, BC is $u(c,\theta) = f(\theta)$. Separation of variables leads to the eigenvalue problem: $\Theta''(\theta) + \lambda\Theta(\theta) = 0$, $\Theta(\theta) = \Theta(\theta + 2\pi)$ and $r^2R'' + rR - \lambda R = 0$, $R_n(r) = r^n$ and $\Theta_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta)$, $\lambda_n = n^2$, $n = 0, 1, 2, \ldots$.

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n (A_n\cos(n\theta) + B_n\sin(n\theta)).$$

Solve for A_n and B_n with the BC, $u(c,\theta) = f(\theta)$. Maximum and minimum temperatures occur on boundary $r = c$.
Fourier Integral, Fourier Cosine and Sine Integrals: Integral representations of functions $f(x)$ on $-\infty < x < \infty$ or $0 < x < \infty$.
Fourier Transform Methods for heat and wave equations: Transforms the x variable to a constant α.
$\mathcal{F}\{u(x,t)\} = U(\alpha,t)$ but does not change the t variable, $\mathcal{F}\{u_t\} = \dfrac{dU}{dt}$ and $\mathcal{F}\{u_{tt}\} = \dfrac{d^2U}{dt^2}$.
Fourier transform on $-\infty < x < \infty$, $\mathcal{F}\{u_{xx}\} = -\alpha^2 U(\alpha,t)$
Fourier Cosine tranform on $0 < x < \infty$, $\mathcal{F}_c\{u_{xx}\} = -\alpha^2 U(\alpha,t) - u_x(0,t)$
Fourier Sine transform on $0 < x < \infty$, $\mathcal{F}_s\{u_{xx}\} = -\alpha^2 U(\alpha,t) + \alpha u(0,t)$

Review for Exam 3: Webwork 5,6, Written Assignments 5,6, Handouts

Practice Problems:

1. Find the steady-state solution to the heat equation, then solve the nonhomogeneous BVP.

   PDE $u_t = u_{xx} + 3x^2 - 1,\ 0 < x < 1,\ 0 < t < \infty$

   BC $u(0,t) = 2,\quad u(1,t) = 1,\quad 0 < t < \infty$

   IC $u(x,0) = 2,\quad 0 < x < 1$

2. (a) Solve for the steady-state temperature, $u(r,\theta)$, on a circular membrane of radius $r = 5$ if the temperature on the boundary is $u(5,\theta) = 5 - 7\sin(4\theta)$.
   (b) What is the maximum temperature on the circular membrane? (c) Minimum temperature?

3. Solve the following problems using Fourier Transforms or Separation of Variables with Series Solutions.

   (a) Solve the heat equation on $0 < x < \infty$.
   PDE $u_t = u_{xx},\ 0 < x < \infty,\ 0 < t < \infty$
   BC $u(0,t) = 0,\ 0 < t < \infty$
   IC $u(x,0) = xe^{-0.2x},\ 0 < x < \infty$.

   (b) Solve for the heat equation on the square $0 < x < 1$, $0 < y < 1$.
   PDE $u_t = k(u_{xx} + u_{yy}),\ 0 < x < 1,\ 0 < y < 1,\ 0 < t < \infty$
   BC $u(0,y,t) = 0,\ u(1,y,t) = 0,\ 0 < y < 1,\ 0 < t < \infty$
   BC $u(x,0,t) = 0,\ u(x,1,t) = 0,\ 0 < x < 1,\ 0 < t < \infty$
   IC $u(x,y,0) = 5\sin(2\pi x)\sin(3\pi y),\ 0 < x < 1,\ 0 < y < 1$

   (c) For the previous problem on the unit square, suppose the initial temperature is $u(x,y,0) = 3$ and all four boundaries are insulated ($u_x(0,y,t) = 0$, $u_y(x,0,t) = 0$ etc.), what is the temperature at all points on the square, $u(x,y,t)$?

   (d) Solve the heat equation with lateral heat loss on $0 < x < \pi$ (be careful about the BCs).
   PDE $u_t = ku_{xx} - 4u,\ 0 < x < \pi,\ 0 < t < \infty$
   BC $u(0,t) = 0,\ u_x(\pi,t) = 0,\ 0 < t < \infty$
   IC $u(x,0) = 2\sin(3x/2),\ 0 < x < \pi$.

   (e) Solve the wave equation on $0 < x < \infty$.
   PDE $u_{tt} = u_{xx},\ 0 < x < \infty,\ 0 < t < \infty$
   BC $u_x(0,t) = 0,\ 0 < t < \infty$
   IC $u(x,0) = 0,\ \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = \begin{cases} 2, & 0 < x < 1 \\ 0, & x > 1. \end{cases}$