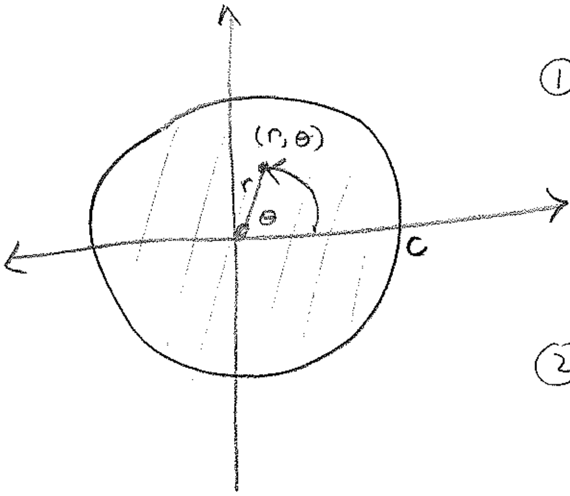
13.2 Wave Equation on a circular membrane



1. We change the coordinates (x,y) to polar coordinates (r,θ).

$$u(x,y,t) \Rightarrow u(r,\theta,t)$$

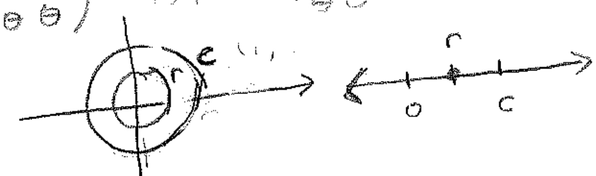
2. We will do the case where the vibrations do not depend on the angular component θ!

$$u(r,\theta,t) \Rightarrow \boxed{u(r,t)}$$

Wave equation in polar coordinates:

$$(*) \quad u_{tt} = a^2\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right) \quad \text{but } u_{\theta\theta} = 0$$

$$(**) \quad \boxed{u_{tt} = a^2\left(u_{rr} + \frac{1}{r}u_r\right)}$$



We will discuss $(*)$ with $u_{\theta\theta} \neq 0$ later.

Use separation of variables for $(**)$,

$$\boxed{u(r,t) = R(r)T(t) = RT}$$

$$\begin{cases} \text{PDE:} & u_{tt} = a^2\left(u_{rr} + \frac{1}{r}u_r\right) \quad 0<r<c,\ t>0 \\ \text{B.C.} & u(c,t)=0,\quad t>0 \\ \text{I.C.} & u(r,0)=f(r),\ \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(r),\quad 0<r<c \end{cases}$$

$$u = RT$$

$$u_{tt} = a^2\left(u_{rr} + \frac{1}{r}u_r\right) \Rightarrow \frac{RT''}{a^2RT} = \frac{a^2}{a^2}\frac{\left(R''T + \frac{1}{r}R'T\right)}{RT}$$

$$\Rightarrow \quad \frac{T''}{a^2T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -\lambda$$

Two ODEs: $\underline{T'' + a^2\lambda T = 0}$ and $\underline{R'' + \frac{1}{r}R' + \lambda R = 0}$

Rewrite the ODE in R to give an eigenvalue problem with the homogeneous B.C. $u(c,t) = 0$

$$(**) \quad \boxed{r^2R'' + rR' + \lambda r^2 R = 0}$$

$$R(c) = 0$$

$$\lim_{r\to 0} R(r) \text{ is bounded}$$

The ODE $(**)$ is a <u>Bessel's Equation of order zero</u>. In general, a <u>Bessel's Equation of order n</u> is

$$\boxed{r^2R'' + rR' + (\underbrace{\lambda r^2}_{\alpha^2 r^2} - n^2)R = 0}$$

provided $\underline{\lambda > 0}$.

For $\underline{\underline{\lambda = \alpha^2}}$, the general solution of $(**)$ is

$$R(r) = \cancel{c_1} J_0(\alpha r) + c_2 \cancel{Y_0}(\alpha r)$$

$J_0(\alpha r)$ and $Y_0(\alpha r)$ are Bessel functions of order zero, first kind, and second kind, respectively.

But $Y_0(\alpha r)$ is <u>not</u> bounded at $r=0$

$$\boxed{R(r) = J_0(\alpha r)}$$

Appy the BC $R(c)=0$ to find the eigenvalues

$$R(c) = J_0(\alpha c) = 0 \text{ for } \alpha_1 < \alpha_2 < \alpha_3 < \dots < \alpha_n < \dots$$

$\boxed{\lambda_n = \alpha_n^2}$ are the eigenvalues and

$\boxed{R_n(r) = J_0(\alpha_n r)}$ are the eigenfunctions,

From Storm-Liouville Theory. These eigenfunctions are orthogonal on $[0,c]$ with respect to the weight function r

$$\int_0^c r\, J_0(\alpha_n r)\, J_0(\alpha_m r)\, dr = 0,\ n \neq m$$

Time equation: $T_n'' + a\alpha_n^2 T = 0$

$$\boxed{T_n(t) = a_n \cos(a\alpha_n t) + b_n \sin(a\alpha_n t)}$$

Product solutions $u_n(r,t) = R_n(r)\, T_n(t)$

$$u(r,t) = \sum_{n=1}^{\infty} u_n(r,t)$$

$$= \sum_{n=1}^{\infty} \underbrace{\left[a_n \cos(a\alpha_n t) + b_n \sin(a\alpha_n t)\right]}_{T_n(t)} \underbrace{J_0(\alpha_n r)}_{R_n(r)}$$

We use properties of Bessel functions to compute α_n, $n = 1, 2, 3, \dots$

Use of Compoters

Solve PDE $u_{tt} = u_{rr} + \frac{1}{r} u_r, \quad 0 < r < 1, \quad t > 0$

B.C $u(1,t) = 0, \quad t > 0$

I.C. $u(r,0) = 0, \quad 0 < r < 1$

I.C. $\left.\frac{\partial u}{\partial t}\right|_{t=0} = \left\{ \begin{array}{ll} -1, & 0 \le r < \frac{1}{4} \\ 0, & \frac{1}{4} < r < 1 \end{array} \right\} = g(r)$

$$\boxed{u(r,t) = \sum_{n=1}^{\infty} \left[a_n \cos(\alpha_n t) + b_n \sin(\alpha_n t)\right] J_0(\alpha_n r)}$$

$u(r,0) =$ ______ Coefficents ______

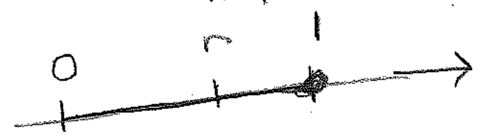
$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} b_n \alpha_n J_0(\alpha_n r) = g(r)$$

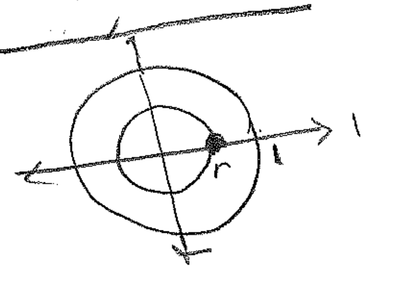
$$b_n \alpha_n = \frac{\int_0^1 r J_0(\alpha_n r) g(r)\, dr}{\int_0^1 r J_0^2(\alpha_n r)\, dr} = \frac{-\int_0^{\frac{1}{4}} r J_0(\alpha_n r)\, dr}{\int_0^1 r J_0^2(\alpha_n r)\, dr}$$

We use the computer to compute α_n and b_n up to $n = 20$, and graph

$u(r,t) \approx \sum_{n=1}^{20}$ ______

t=0





Generalization of the Wave Equation

$u(r,\theta,t)$

PDE $u_{tt} = a^2\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right)$ $\quad 0<r<c$

$0<\theta<2\pi$

$t>0$

B.C. $u(c,\theta,t)=0$.

$u(c,\theta,t) = u(c,\theta+2\pi,t)$.

I.C. $u(r,\theta,0) = f(r,\theta)$

$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(r,\theta)$

$$\boxed{u(r,\theta,t) = R(r)\,\Theta(\theta)T(t)}$$

$$\begin{cases} \Theta'' + \nu\Theta = 0 & \Theta(\theta) = \Theta(\theta+2\pi) \\ r^2R'' + rR' + (\lambda r^2 - \underbrace{\nu}_{n^2})R = 0, & R(c)=0 \\ T'' + a^2\lambda T = 0 & \end{cases}$$

$\nu = n^2$, $\Theta_n = A_n\cos(n\theta) + B_n\sin(n\theta)$ $n=1,2,\ldots$

$\lambda_{n,m} = \alpha_{n,m}^2$, $R_{n,m} = J_n(\alpha_{n,m}r)$, $n=1,2,\ldots$ $m=1,2,\ldots$

$T_{n,m} = C_{n,m}\cos(a\alpha_{n,m}t) + D_{nm}\sin(a\alpha_{n,m}t)$

$n=1,2,\ldots$

$m=1,2,\ldots$

$$u(r,\theta,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} T_{n,m}(t)\,R_{n,m}(r)\,\Theta_n(\theta)$$

See problem #17 p. 514

<u>Question:</u> If the initial position of the circular membrane is $\boxed{u(r,\theta,0)=0}$, write the simplified form of the solution to the wave equation.

$$u(r,\theta,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\underbrace{\left[C_{n,m}\cos(a\alpha_{n,m}t)+D_{n,m}\sin(a\alpha_{n,m}t)\right]}_{T_{nm}}\underbrace{J_n(\alpha_{n,m}r)}_{R_{n,m}}\underbrace{\left[A_n\cos(n\theta)+B_n\sin(n\theta)\right]}_{\Theta_n}$$

$u(r,\theta,0)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}$ ________________________________

Coefficients ________________________

$u(r,\theta,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}$ ________________________________

Laplace Transform of $f(t)$ for $t \in [0,\infty)$.

$$* \quad \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\, 1\, dt = \frac{1}{s}$$

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\, dt = \frac{1}{s-a}$$

$$\mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2}$$

$$\mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}$$

We will use the identities for the derivative and apply Laplace transforms to solve PDEs with time derivatives.
$u(x,t)$

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) = sF(s) - f(0)$$

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

$$= s^2F(s) - sf(0) - f'(0)$$

Example $\mathcal{L}\{10 - 3\cos(2t)\} =$ ____________

$\mathcal{L}^{-1}\left\{\frac{4}{s} + \frac{51}{s^2+4}\right\} =$ ____________