# Game theory and epidemiology

## Biomathematics Seminar Fall 2016

Fan Bai

Department of Mathematics and Statistics, Texas Tech University

September 13, 2016

# Game theory

1. Game theory is the formal study on "how to choose", based on maximizing players' payoff. Game theory has been used in many disciplines such as economics, political science, business, information science, and biology as well.

# Game theory

1. Game theory is the formal study on "how to choose", based on maximizing players' payoff. Game theory has been used in many disciplines such as economics, political science, business, information science, and biology as well.
2. Game theory has been applied to predict human behavior in the context of epidemiology for over two decades. People choose the best strategy to maximize their own payoffs, based on the outcomes of different strategies.

# Game theory

1. Game theory is the formal study on "how to choose", based on maximizing players' payoff. Game theory has been used in many disciplines such as economics, political science, business, information science, and biology as well.
2. Game theory has been applied to predict human behavior in the context of epidemiology for over two decades. People choose the best strategy to maximize their own payoffs, based on the outcomes of different strategies.
3. In epidemiology, we actually mean "vaccination". Since this is the scenario when people need to choose.

# Game theory

1. Game theory is the formal study on "how to choose", based on maximizing players' payoff. Game theory has been used in many disciplines such as economics, political science, business, information science, and biology as well.
2. Game theory has been applied to predict human behavior in the context of epidemiology for over two decades. People choose the best strategy to maximize their own payoffs, based on the outcomes of different strategies.
3. In epidemiology, we actually mean "vaccination". Since this is the scenario when people need to choose.
4. The theory of "Game theory and vaccination" is identical with evolutionary game theory. These two both deal with problems on population-levels.

# Example

We consider a population with two types.

| Example | | |
|---|---|---|
| | Type $A$ | Type $B$ |
| Type $A$ | $a$ | $b$ |
| Type $B$ | $c$ | $d$ |

This payoff matrix is used to describe interactions of two types: If $A$ interacts with another $A$, the payoff is $a$; and so on. We have

$$x = x_1 = 1 - x_2,$$

where $x_1$ is the fraction of type $A$ and $x_2$ is the fraction of type $B$. The payoffs are

$$\begin{aligned} \pi_A &= ax + b(1-x), \\ \pi_B &= cx + d(1-x). \end{aligned}$$

The dynamics of this type of game is described by replicator equation (General formula):

$$\dot{x}_i = x_i(\pi_i - \langle \pi \rangle).$$

$\langle \pi \rangle$ represents the average payoff of the whole population. In this game, the replicator equation is

$$\dot{x} = x(1 - x)[(a - b - c + d)x + b - d].$$

There are three fixed points, two trivial ones are $x = 0$ and $x = 1$. The third one is

$$x^{\star} = \frac{d - b}{a - b - c + d},$$

for $a > c$ and $d > b$ or for $a < c$ and $d < b$.

1. *Dominance*. If $a > c$ and $b > d$, type $A$ dominates type $B$. In this case, the fixed point at $x = 1$ is stable and the fixed point at $x = 0$ is unstable. If $a < c$ and $b < d$, type $B$ dominates type $A$. In this case, the fixed point at $x = 0$ is stable and the fixed point at $x = 1$ is unstable.
2. *Bistability*. If $a > c$ and $d > b$, the fixed points at $x = 0$ and $x = 1$ are both stable and the fixed point at $x^\star$ is unstable.
3. *Coexistence*. If $a < c$ and $d < b$, the fixed points at $x = 0$ and $x = 1$ are both unstable and the fixed point at $x^\star$ is stable. The population eventually becomes a stable mixture of type $A$ and type $B$. This is the case we will meet in the vaccination games.
4. *Neutrality*. For $a = c$ and $b = d$.

# The formal game and its payoff matrix

Vaccination offers protection, but also cost and risk. Before the outbreak of epidemic, people need to evaluate the costs of infection and vaccination. Eventually, a certain percentage of people goes into the vaccinated class, while others stay in the susceptible class.
The formal vaccination game can be described as:

| Vaccination Game as general | | |
|---|---|---|
| | Vaccinated individual $j$ | Susceptible individual $j$ |
| Vaccinated individual $i$ | $-C_v$ | $-C_v$ |
| Susceptible individual $i$ | 0 | $-\pi_p C_i$ |

All elements can be translated easily. By using game theory, the expected vaccine coverage level is expressed in terms of the attack ratio. We need to use mathematical models to find another relation between attack ratio and vaccine coverage level, to make the prediction more accurate.

# Flowchart

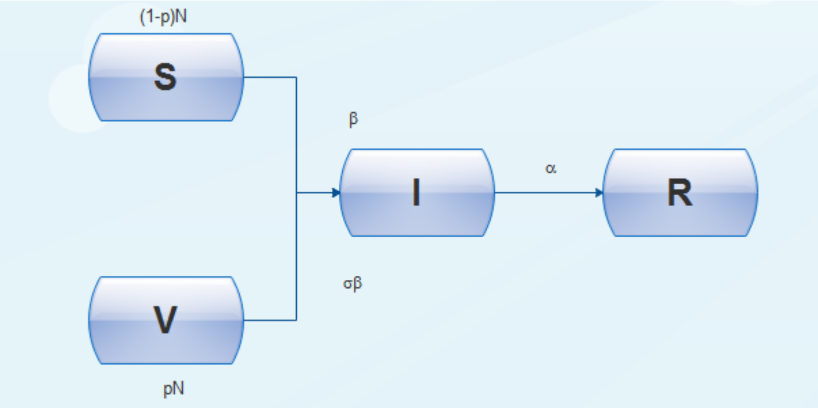


**Figure:** Vaccination model

The major goal of "Vaccination games" is to predict the expected vaccine coverage levels. Two weapons are: Game theory and mathematical models.

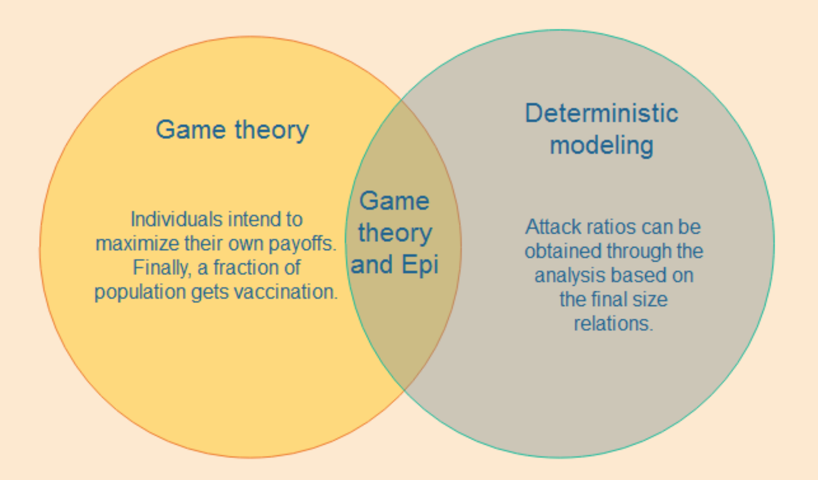


**Figure:** Vaccination game theory

# References


- von Neumann J. and Morgenstern O., 1944, *Theory of Games and Economic Behavior*, Princeton University Press, Princeton.
- Maynard S. J. and Price G.R., 1973, *The logic of animal conflicts*, *Nature* 246:15-18.
- Nowak M. A. and Sigmund K., 2004, *Evolutionary dynamics of biological games*, *Science* 303:793-799.
- Fine P. E. M., Clarkson J. A., 1986, *Individual versus public priorities in the determination of optimal vaccination policies*, *American Journal of Epidemiology* 124: 1012-1020.
- Tassier T., 2013, *The Economics of Epidemiology*, Springer.
- Bauch C.T., Earn D.J.D., 2004, *Vaccination and the theory of games*, *Proc. Natl Acad. Sci. USA* 101: 13391-13394.

# Vaccination game with perfect vaccine

| Fully-effective Vaccination Game | | |
|---|---|---|
| | Vaccinated individual *j* | Susceptible individual *j* |
| Vaccinated individual *i* | $-C_v$ | $-C_v$ |
| Susceptible individual *i* | 0 | $-\pi_p C_i$ |

# Vaccination game with perfect vaccine

| Fully-effective Vaccination Game | | |
|---|---|---|
| | Vaccinated individual $j$ | Susceptible individual $j$ |
| Vaccinated individual $i$ | $-C_v$ | $-C_v$ |
| Susceptible individual $i$ | 0 | $-\pi_p C_i$ |

1. Cost of infection is $-C_i$ and cost of vaccination is $-C_v$.

# Vaccination game with perfect vaccine

| Fully-effective Vaccination Game | | |
|---|---|---|
| | Vaccinated individual $j$ | Susceptible individual $j$ |
| Vaccinated individual $i$ | $-C_v$ | $-C_v$ |
| Susceptible individual $i$ | 0 | $-\pi_p C_i$ |

1. Cost of infection is $-C_i$ and cost of vaccination is $-C_v$.
2. All elements are negative.

# Vaccination game with perfect vaccine

| Fully-effective Vaccination Game | | |
|---|---|---|
| | Vaccinated individual $j$ | Susceptible individual $j$ |
| Vaccinated individual $i$ | $-C_v$ | $-C_v$ |
| Susceptible individual $i$ | 0 | $-\pi_p C_i$ |

1. Cost of infection is $-C_i$ and cost of vaccination is $-C_v$.
2. All elements are negative.
3. We assume that $C_i > C_v$.

# Vaccination game with perfect vaccine

| Fully-effective Vaccination Game | | |
|---|---|---|
| | Vaccinated individual $j$ | Susceptible individual $j$ |
| Vaccinated individual $i$ | $-C_v$ | $-C_v$ |
| Susceptible individual $i$ | 0 | $-\pi_p C_i$ |

1. Cost of infection is $-C_i$ and cost of vaccination is $-C_v$.
2. All elements are negative.
3. We assume that $C_i > C_v$.
4. NE (Nash equilibrium) is: fraction of population to be vaccinated is $p$.

# Vaccination game with perfect vaccine

| Fully-effective Vaccination Game | | |
|---|---|---|
| | Vaccinated individual $j$ | Susceptible individual $j$ |
| Vaccinated individual $i$ | $-C_v$ | $-C_v$ |
| Susceptible individual $i$ | 0 | $-\pi_p C_i$ |

1. Cost of infection is $-C_i$ and cost of vaccination is $-C_v$.
2. All elements are negative.
3. We assume that $C_i > C_v$.
4. NE (Nash equilibrium) is: fraction of population to be vaccinated is $p$.
5. Question is: How to find $p$?

Assume the final fraction of people who take vaccination is $p$; the fraction of people to be vaccinated is $x_v$ and the fraction of people not to be vaccinated is $x_s$, $\{x_v, x_s\} = \{p, 1-p\}$.

Assume the final fraction of people who take vaccination is $p$; the fraction of people to be vaccinated is $x_v$ and the fraction of people not to be vaccinated is $x_s$, $\{x_v, x_s\} = \{p, 1-p\}$.

1.
$$f_v = -C_v x_v - C_v x_s = -C_v,$$
$$f_s = -\pi_p C_i x_s.$$

Assume the final fraction of people who take vaccination is $p$; the fraction of people to be vaccinated is $x_v$ and the fraction of people not to be vaccinated is $x_s$, $\{x_v, x_s\} = \{p, 1-p\}$.

1.
$$f_v = -C_v x_v - C_v x_s = -C_v,$$
$$f_s = -\pi_p C_i x_s.$$

2.
$$\dot{x_v} = x_v(f_v - \bar{f}),$$
$$\dot{x_s} = x_s(f_s - \bar{f}),$$

with
$$\bar{f} = x_v f_v + x_s f_s.$$

Assume the final fraction of people who take vaccination is $p$; the fraction of people to be vaccinated is $x_v$ and the fraction of people not to be vaccinated is $x_s$, $\{x_v, x_s\} = \{p, 1-p\}$.

1. 
$$f_v = -C_v x_v - C_v x_s = -C_v,$$
$$f_s = -\pi_p C_i x_s.$$

2. 
$$\dot{x_v} = x_v(f_v - \bar{f}),$$
$$\dot{x_s} = x_s(f_s - \bar{f}),$$
with
$$\bar{f} = x_v f_v + x_s f_s.$$

3. The fraction $x_v = 1 - \frac{C_v}{\pi_p C_i}$. If we define the relative cost $r = \frac{C_v}{C_i}$, the expected vaccine coverage level is:
$$p = 1 - \frac{r}{\pi_p}.$$

Assume the final fraction of people who take vaccination is $p$; the fraction of people to be vaccinated is $x_v$ and the fraction of people not to be vaccinated is $x_s$, $\{x_v, x_s\} = \{p, 1-p\}$.

1. $$f_v = -C_v x_v - C_v x_s = -C_v,$$
   $$f_s = -\pi_p C_i x_s.$$
2. $$\dot{x_v} = x_v(f_v - \bar{f}),$$
   $$\dot{x_s} = x_s(f_s - \bar{f}),$$
   with
   $$\bar{f} = x_v f_v + x_s f_s.$$
3. The fraction $x_v = 1 - \frac{C_v}{\pi_p C_i}$. If we define the relative cost $r = \frac{C_v}{C_i}$, the expected vaccine coverage level is:
   $$p = 1 - \frac{r}{\pi_p}.$$
4. $\pi_p$ can be estimated by epidemic models.

1. The SIR mathematical model is

$$\frac{dS}{dt} = -\beta SI$$
$$\frac{dI}{dt} = \beta SI - \alpha I$$
$$\frac{dR}{dt} = \alpha I,$$

1. The SIR mathematical model is

$$\frac{dS}{dt} = -\beta SI$$
$$\frac{dI}{dt} = \beta SI - \alpha I$$
$$\frac{dR}{dt} = \alpha I,$$

2. The final size relation can be obtained from SIR model,

$$\ln \frac{S_0}{S_\infty} = \frac{\beta}{\alpha}[S_0 - S_\infty].$$

1. The SIR mathematical model is

$$\frac{dS}{dt} = -\beta SI$$
$$\frac{dI}{dt} = \beta SI - \alpha I$$
$$\frac{dR}{dt} = \alpha I,$$

2. The final size relation can be obtained from SIR model,

$$\ln \frac{S_0}{S_\infty} = \frac{\beta}{\alpha}[S_0 - S_\infty].$$

3. The attack ratio can be expressed in terms of $p$,

$$\ln \frac{1}{1 - \pi_p} = \frac{\beta}{\alpha}(1 - p)N\pi_p.$$

# Vaccination with imperfect vaccine

| Partial-effective Vaccination Game | | |
|---|---|---|
| | Vaccinated individual | Susceptible individual |
| Vaccinated individual | $-C_v - \sigma \pi_v C_i$ | $-C_v - \sigma \pi_p C_i$ |
| Susceptible individual | $-\pi_v C_i$ | $-\pi_p C_i$ |

# Vaccination with imperfect vaccine

| Partial-effective Vaccination Game | | |
|---|---|---|
| | Vaccinated individual | Susceptible individual |
| Vaccinated individual | $-C_v - \sigma\pi_v C_i$ | $-C_v - \sigma\pi_p C_i$ |
| Susceptible individual | $-\pi_v C_i$ | $-\pi_p C_i$ |

1. Infection factor $\sigma$, $\sigma = 0$ represents the vaccine is perfect.

# Vaccination with imperfect vaccine

| Partial-effective Vaccination Game | | |
|---|---|---|
| | Vaccinated individual | Susceptible individual |
| Vaccinated individual | $-C_v - \sigma\pi_v C_i$ | $-C_v - \sigma\pi_p C_i$ |
| Susceptible individual | $-\pi_v C_i$ | $-\pi_p C_i$ |

1. Infection factor $\sigma$, $\sigma = 0$ represents the vaccine is perfect.
2. Cost of infection is $-C_i$ and cost of vaccination is $-C_v$, $\pi_v$ and $\pi_p$ are attack ratios.

# Vaccination with imperfect vaccine

| Partial-effective Vaccination Game | | |
|---|---|---|
| | Vaccinated individual | Susceptible individual |
| Vaccinated individual | $-C_v - \sigma\pi_v C_i$ | $-C_v - \sigma\pi_p C_i$ |
| Susceptible individual | $-\pi_v C_i$ | $-\pi_p C_i$ |

1. Infection factor $\sigma$, $\sigma = 0$ represents the vaccine is perfect.
2. Cost of infection is $-C_i$ and cost of vaccination is $-C_v$, $\pi_v$ and $\pi_p$ are attack ratios.
3. The expected vaccine coverage level $p$ and attack ratios $\pi_p$ and $\pi_v$ satisfy

$$\frac{r}{1-\sigma} = p\pi_v + (1-p)\pi_p,$$

with $r$ relative cost to measure the vaccine. $\pi_p$ and $\pi_v$ can be expressed by SVIR model.

# Vaccination with imperfect vaccine

| Partial-effective Vaccination Game | | |
|---|---|---|
| | Vaccinated individual | Susceptible individual |
| Vaccinated individual | $-C_v - \sigma\pi_v C_i$ | $-C_v - \sigma\pi_p C_i$ |
| Susceptible individual | $-\pi_v C_i$ | $-\pi_p C_i$ |

1. Infection factor $\sigma$, $\sigma = 0$ represents the vaccine is perfect.
2. Cost of infection is $-C_i$ and cost of vaccination is $-C_v$, $\pi_v$ and $\pi_p$ are attack ratios.
3. The expected vaccine coverage level $p$ and attack ratios $\pi_p$ and $\pi_v$ satisfy

$$\frac{r}{1-\sigma} = p\pi_v + (1-p)\pi_p,$$

   with $r$ relative cost to measure the vaccine. $\pi_p$ and $\pi_v$ can be expressed by SVIR model.
4. $p$ is smaller than the threshold value $p_c$ (Herd immunity threshold).

1. With replicator equation, we have:

$$\frac{r}{1-\sigma} = p\pi_V + (1-p)\pi_p.$$

1. With replicator equation, we have:

$$\frac{r}{1-\sigma} = p\pi_V + (1-p)\pi_P.$$

2. Construct the SVIR model,

$$\frac{dS}{dt} = -\beta SI,$$
$$\frac{dV}{dt} = -\sigma\beta VI,$$
$$\frac{dI}{dt} = \beta SI + \sigma\beta VI - \alpha I,$$
$$\frac{dR}{dt} = \alpha I,$$

1. With replicator equation, we have:

$$\frac{r}{1-\sigma} = p\pi_V + (1-p)\pi_p.$$

2. Construct the SVIR model,

$$\frac{dS}{dt} = -\beta SI,$$
$$\frac{dV}{dt} = -\sigma\beta VI,$$
$$\frac{dI}{dt} = \beta SI + \sigma\beta VI - \alpha I,$$
$$\frac{dR}{dt} = \alpha I,$$

3. The final size relation can be derived from the model,

$$\ln\frac{S_\infty}{S_0} = \frac{\beta}{\alpha}[S_\infty + V_\infty - S_0 - V_0],$$
$$\ln\frac{V_\infty}{V_0} = \frac{\sigma\beta}{\alpha}[S_\infty + V_\infty - S_0 - V_0].$$

Epi-game theory has been applied to different types of epidemic:

- Influenza
- Smallpox
- Chickenpox
- Measles
- Rubella

# Uniqueness

$$\pi_p(1-p) = r,$$

$$p\pi_v + (1-p)\pi_p = \frac{r}{1-\sigma}.$$

For uniqueness:

1. From the equations of vaccine coverage level in vaccination game with perfect vaccine, to differentiate the left side of the equation,

$$\pi_p'(1-p) - \pi_p,$$

2. From the equations of vaccine coverage level in vaccination game with imperfect vaccine, to differentiate the left side of the equation,

$$\begin{aligned} & \pi_v + p\pi_v' - \pi_p + (1-p)\pi_p' \\ = \quad & (\pi_v - \pi_p) + p\pi_v' + (1-p)\pi_p'. \end{aligned}$$

# Uniqueness

$$\pi_p(1-p) = r,$$

$$p\pi_v + (1-p)\pi_p = \frac{r}{1-\sigma}.$$

For uniqueness:

1. From the equations of vaccine coverage level in vaccination game with perfect vaccine, to differentiate the left side of the equation,

$$\pi_p'(1-p) - \pi_p,$$

2. From the equations of vaccine coverage level in vaccination game with imperfect vaccine, to differentiate the left side of the equation,

$$\begin{aligned} & \pi_v + p\pi_v' - \pi_p + (1-p)\pi_p' \\ = \quad & (\pi_v - \pi_p) + p\pi_v' + (1-p)\pi_p'. \end{aligned}$$

# Uniqueness

$$\pi_p(1-p) = r,$$

$$p\pi_v + (1-p)\pi_p = \frac{r}{1-\sigma}.$$

For uniqueness:

1. From the equations of vaccine coverage level in vaccination game with perfect vaccine, to differentiate the left side of the equation,

$$\pi_p'(1-p) - \pi_p,$$

2. From the equations of vaccine coverage level in vaccination game with imperfect vaccine, to differentiate the left side of the equation,

$$\begin{aligned} & \pi_v + p\pi_v' - \pi_p + (1-p)\pi_p' \\ = \quad & (\pi_v - \pi_p) + p\pi_v' + (1-p)\pi_p'. \end{aligned}$$

3. Both right sides of two equations are constants. $\pi_p' < 0$ and $\pi_v' < 0$ are important for uniqueness of Nash equilibria.

# Perfect vaccine

$$\ln \frac{S_0}{S_\infty} = \frac{\beta}{\alpha} S_0 [1 - \frac{S_\infty}{S_0}],$$

$$\frac{dS_\infty}{dp} = -\frac{\frac{1}{S_0} - \frac{\beta}{\alpha}}{\frac{1}{S_\infty} - \frac{\beta}{\alpha}} > 0,$$

$$\frac{d\pi_p}{dp} < 0.$$

## Theorem

*In the SIR compartmental model, when vaccine coverage level $p$ increases from $0$ to the herd immunity threshold $p_c$, the final size $S_\infty$ of susceptible class will increase, the attack ratio will decrease. If $p \geq p_c$, the whole population is protected completely by the vaccine.*

# Imperfect vaccine

## Lemma

*In the SVIR model with infection factor $\sigma$, there exists a threshold value $\sigma_c$, which is defined as*

$$\left(\frac{S_0}{S_\infty}\right)^{\sigma_c} = \mathcal{R}_c = \frac{\beta}{\alpha} N.$$

*If the infection factor $\sigma$ is smaller than $\sigma_c$, $\frac{V_0}{V_\infty} < \mathcal{R}_c < \frac{S_0}{S_\infty}$ holds when the vaccine coverage level is low, and by increasing the vaccine coverage level p, $\mathcal{R}_c < \frac{V_0}{V_\infty} < \frac{S_0}{S_\infty}$. If the infection factor $\sigma$ is bigger than $\sigma_c$, the inequality $\mathcal{R}_c < \frac{V_0}{V_\infty} < \frac{S_0}{S_\infty}$ holds.*

## Theorem

*If $\sigma$ is smaller than $\sigma_c$, there exists a critical value $p_0$, as vaccine coverage level p increasing from 0 to $p_0$, $S_\infty$ will decrease and $V_\infty$ will still increase; while p increasing from $p_0$ to 1, $S_\infty$ and $V_\infty$ both increase, where $p_0$ is the critical value we described in previous lemma. If $\sigma$ is bigger than $\sigma_c$, when the vaccine coverage level p increases, $S_\infty$ will decrease and $V_\infty$ will increase. If the vaccine coverage level p increases between 0 and the herd immunity threshold, for vaccinated group, for non-vaccinated group and for the whole population, the attack ratios decrease. The infection factor $\sigma$ and the attack ratios are independent.*

# application of age-of-infection models

1. If in previous two vaccination games, the diseases have more compartments such as exposed stage, whether the results on attack ratios still hold? Whether these two games have unique Nash equilibria?

# application of age-of-infection models

1. If in previous two vaccination games, the diseases have more compartments such as exposed stage, whether the results on attack ratios still hold? Whether these two games have unique Nash equilibria?
2. All compartmental epidemic models can be describe by age-of-infection model, such as:

$$S' = -\frac{a}{N_0} S\phi$$

$$\begin{aligned}\phi(t) &= \phi_0(t) + \int_0^t \tfrac{a}{N_0} S(t-\tau)\phi(t-\tau)A(\tau)d\tau \\ &= \phi_0(t) + \int_0^t [-S'(t-\tau)]A(\tau)d\tau.\end{aligned}$$

We are able to prove the final size relation to this general age-of-infection model is similar comparing with SIR/SVIR models.

# application of age-of-infection models

1. If in previous two vaccination games, the diseases have more compartments such as exposed stage, whether the results on attack ratios still hold? Whether these two games have unique Nash equilibria?
2. All compartmental epidemic models can be describe by age-of-infection model, such as:

$$S' = -\frac{a}{N_0} S\phi$$

$$\begin{aligned} \phi(t) &= \phi_0(t) + \int_0^t \tfrac{a}{N_0} S(t-\tau)\phi(t-\tau)A(\tau)d\tau \\ &= \phi_0(t) + \int_0^t [-S'(t-\tau)]A(\tau)d\tau. \end{aligned}$$

   We are able to prove the final size relation to this general age-of-infection model is similar comparing with SIR/SVIR models.
3. One example: influenza has exposed stage. We analyze the vaccination game of influenza, this game has one unique Nash equilibrium.

1. If the epidemic is described by SEIR model,

$$
\begin{aligned}
S' &= -\beta S(I + \epsilon E) \\
E' &= \beta S(I + \epsilon E) - \kappa E \\
I' &= \kappa E - \alpha I \\
R' &= \alpha I.
\end{aligned}
$$

1. If the epidemic is described by SEIR model,

$$\begin{aligned} S' &= -\beta S(I + \epsilon E) \\ E' &= \beta S(I + \epsilon E) - \kappa E \\ I' &= \kappa E - \alpha I \\ R' &= \alpha I. \end{aligned}$$

2. This SEIR can be expressed by age-of-infection model with distribution function

$$A(\tau) = \epsilon e^{-\kappa\tau} + \frac{\kappa}{\kappa - \alpha}[e^{-\alpha\tau} - e^{-\kappa\tau}].$$

1. If the epidemic is described by SEIR model,

$$\begin{aligned} S' &= -\beta S(I + \epsilon E) \\ E' &= \beta S(I + \epsilon E) - \kappa E \\ I' &= \kappa E - \alpha I \\ R' &= \alpha I. \end{aligned}$$

2. This SEIR can be expressed by age-of-infection model with distribution function

$$A(\tau) = \epsilon e^{-\kappa\tau} + \frac{\kappa}{\kappa - \alpha}[e^{-\alpha\tau} - e^{-\kappa\tau}].$$

3. The final size relation of age-of-infection model shows that the attack ratios are decreasing functions, vaccination games with SEIR-structure epidemics have unique NEs.

Bauch C.T., Earn D.J.D., 2004, *Vaccination and the theory of games*, *Proc. Natl Acad. Sci. USA* 101: 13391-13394.

Bai F., 2016, *Uniqueness of Nash equilibrium in vaccination games*, *Journal of Biological Dynamics* 10: 395-415.

# formal games

Some basic setup:

1, The sizes of two sub-groups are $N_1$ and $N_2$, respectively. $N_1$ and $N_2$ are constants.

2, Group $i$ members make $a_i$ contacts in unit time and the fraction of contacts made by a member of group $i$ is with a member of group $j$ is $p_{ij}$ ,$(i, j = 1, 2)$.

3, Costs of vaccination for members in sub-group 1 is $C_{v1}$ and for members in sub-group 2 is $C_{v2}$.

4, Probabilities of being infected for two groups are $\pi_{p1}$ and $\pi_{q2}$ associated with the vaccine coverage level $p$ and $q$ in each subgroup.

# Payoff matrices

| Fully-effective vaccination game with two-subgroups | | | | |
|---|---|---|---|---|
| | Vaccinated individual 1 | Susceptible individual 1 | Vaccinated individual 2 | Susceptible individual 2 |
| Vaccinated individual 1 | $-C_{v1}$ | $-C_{v1}$ | $-C_{v1}$ | $-C_{v1}$ |
| Susceptible individual 1 | 0 | $-\pi_{p1} C_{i1}$ | 0 | $-\pi_{q2} C_{i1}$ |
| Vaccinated individual 2 | $-C_{v2}$ | $-C_{v2}$ | $-C_{v2}$ | $-C_{v2}$ |
| Susceptible individual 2 | 0 | $-\pi_{p1} C_{i2}$ | 0 | $-\pi_{q2} C_{i2}$ |

The NE can be expressed as:

$$p = 1 - \frac{C_{r1} p_{22} - C_{r2} p_{21}}{p_{11} p_{22} - p_{12} p_{21}} \frac{1}{\pi_{p1}}, \quad q = 1 - \frac{C_{r2} p_{11} - C_{r1} p_{12}}{p_{11} p_{22} - p_{12} p_{21}} \frac{1}{\pi_{q2}},$$

| Partial-effective Vaccination Game with two-subgroups | | | | |
|---|---|---|---|---|
| | Vaccinated individual 1 | Susceptible individual 1 | Vaccinated individual 2 | Susceptible individual 2 |
| Vaccinated individual 1 | $-C_{v1} - \sigma_1 \pi_{pv1} C_{i1}$ | $-C_{v1} - \sigma_1 \pi_{pv1} C_{i1}$ | $-C_{v1} - \sigma_1 \pi_{qv2} C_{i1}$ | $-C_{v1} - \sigma_1 \pi_{qi2} C_{i1}$ |
| Susceptible individual 1 | $-\pi_{pv1} C_{i1}$ | $-\pi_{pi1} C_{i1}$ | $-\pi_{qv2} C_{i1}$ | $-\pi_{qi2} C_{i1}$ |
| Vaccinated individual 2 | $-C_{v2} - \sigma_2 \pi_{pv1} C_{i2}$ | $-C_{v2} - \sigma_2 \pi_{pi1} C_{i2}$ | $-C_{v2} - \sigma_2 \pi_{qv2} C_{i2}$ | $-C_{v2} - \sigma_2 \pi_{qi2} C_{i2}$ |
| Susceptible individual 2 | $-\pi_{pv1} C_{i2}$ | $-\pi_{pi1} C_{i2}$ | $-\pi_{qv2} C_{i2}$ | $-\pi_{qi2} C_{i2}$ |

The NE is:

$$p = \frac{\frac{C_{r1}(p_{11}p_{22}+p_{12}p_{22})}{1-\sigma_1} - \frac{C_{r2}(p_{12}p_{21}+p_{12}p_{22})}{1-\sigma_2} - \pi_{pi1}(p_{11}p_{22} - p_{12}p_{21})}{(\pi_{pv1} - \pi_{pi1})(p_{11}p_{22} - p_{12}p_{21})},$$

$$q = \frac{\frac{C_{r1}(p_{11}p_{21}+p_{12}p_{21})}{1-\sigma_1} - \frac{C_{r2}(p_{11}p_{21}+p_{11}p_{22})}{1-\sigma_2} - \pi_{qi2}(p_{12}p_{21} - p_{11}p_{22})}{(\pi_{qv2} - \pi_{qi2})(p_{12}p_{21} - p_{11}p_{22})},$$

# Results on uniqueness

It is quite difficult to prove the uniqueness of vaccine coverage levels $p$ and $q$, the idea is to focus on the properties of several attack ratio functions. This problem is still open.

# References


- Chen F.H., 2006, *A susceptible-infected epidemic model with voluntary vaccination*, *Journal of Mathematical Biology* 53: 253-272.
- Reluga T.C., 2008, *An SIS epidemiology game with two subpopulations*, *J Biol Dyn.* 3: 515-531.
- Bai F., 2016, *Uniqueness of Nash equilibrium in vaccination games*, *Journal of Biological Dynamics* 10: 395-415.

Some interesting and open problems:

1. Long time scale. Involve human birth and human death (Malaria for example).

Some interesting and open problems:

1. Long time scale. Involve human birth and human death (Malaria for example).
2. Combine game theory and stochastic modeling (network modeling).

Some interesting and open problems:

1. Long time scale. Involve human birth and human death (Malaria for example).
2. Combine game theory and stochastic modeling (network modeling).
3. Analysis of uniqueness of NE in vaccination games in population with two or more sub-groups.

Thank you for attending!