Answers ① $u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}\, e^{-(m^2+n^2)t} \sin(mx)\sin(ny)$

$u(x,y,0)=10 \Rightarrow A_{mn}=\frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} 10\cdot \sin(mx)\sin(ny)\,dx\,dy$

$A_{mn}=\frac{40}{\pi^2}\int_0^{\pi}\sin(mx)\,dx\int_0^{\pi}\sin(ny)\,dy$

$=\frac{40}{\pi^2}\left(-\frac{\cos(mx)}{m}\Big|_0^{\pi}\right)\left(-\frac{\cos(ny)}{n}\Big|_0^{\pi}\right)$

$=\frac{40}{\pi^2}\left(\frac{1}{m}-\frac{\cos(m\pi)}{m}\right)\left(\frac{1}{n}-\frac{\cos(n\pi)}{n}\right)$

$$\boxed{u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{40}{\pi^2}\left(\frac{1-\cos(m\pi)}{m}\right)\left(\frac{1-\cos(n\pi)}{n}\right)e^{-(m^2+n^2)t}\sin(mx)\sin(ny)}$$

② $u(x,y,t)=A_{00}+\sum_{m=1}^{\infty}A_{m0}e^{-m^2t}\cos(mx)+\sum_{n=1}^{\infty}A_{0n}e^{-n^2t}\cos(ny)$

$+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}A_{mn}e^{-(m^2+n^2)t}\cos(mx)\cos(ny)$

$u(x,y,0)=A_{00}+\sum_{m=1}^{\infty}A_{m0}\cos(mx)+\sum_{n=1}^{\infty}A_{0n}\cos(ny)$

$+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}A_{mn}\cos(mx)\cos(ny)$

$=10 \quad \Rightarrow A_{00}=10$ and $A_{m0}=0$, $A_{0n}=0$ and $A_{mn}=0$

$$\boxed{u(x,y,t)=10}$$

③ Use form in ① since BC are zero

$u(x,y,0)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}A_{mn}\sin(mx)\sin(ny)$

$=2\sin(x)\sin y \Rightarrow A_{11}=2 \quad A_{mn}=0,\ mn\neq 11$

$$\boxed{u(x,y,t)=2e^{-(1^2+4^2)t}\sin(x)\sin(4y)=2e^{-17t}\sin(x)\sin(4y)}$$

④ Use form in ② since BC are zero flux

$u(x,y,0)=1-\cos(3x)+7\cos(x)\cos(2y)$

$$\boxed{u(x,y,t)=1-e^{-9t}\cos(3x)+7e^{-5t}\cos(x)\cos(2y)}$$