Review Topics for Exam # 1
Math 4354, 8.1-8.2, 10.1-10.3, 11.1-11.3

Review: All material on our website. WeBWorK #1,2, Homework #1,2.

Theory Linear Systems of ODEs: $\vec{X}'(t) = A(t)\vec{X}(t) + \vec{F}(t)$ has general solution: $\vec{X}(t) = \vec{X}_c(t) + \vec{X}_p(t)$, where $\vec{X}_c(t) = c_1\vec{X}_1(t) + c_2\vec{X}_2(t) + \cdots + c_n\vec{X}_n(t)$ is the solution of the homogeneous DE, n linearly independent vector solutions, and $\vec{X}_p(t)$ is any particular solution of the nonhomogeneous DE. The Wronskian is nonzero $W(\vec{X}_1(t), \ldots, \vec{X}_n(t)) \neq 0$ for some t, if the set is independent.
Homogeneous Linear Systems of ODEs: $\vec{X}'(t) = A\vec{X}(t)$. A is a 2×2 matrix. Compute the general solution $c_1\vec{X}_1(t) + c_2\vec{X}_2(t)$ and solve for c_1 and c_2 given an initial vector. (See website handout).The solution depends on the eigenvalues ($\lambda_{1,2}$ real, repeated, or complex) and corresponding eigenvectors ($\vec{K}_{1,2}$ nonzero vectors) of matrix A. Eigenvalues of an upper or a lower triangular matrix A are values on the diagonal.
Stability in Homogeneous Linear Systems of ODEs: $\vec{X}'(t) = A\vec{X}(t)$. The origin (0,0) is a unique critical point iff $|A| \neq 0$. Classify the origin as stable or unstable node, stable or unstable spiral, center or saddle
Nonlinear Autonomous Systems of ODE: $dx/dt = P(x, y)$, $dy/dt = Q(x, y)$. Critical points are computed by setting the derivatives equal to zero, $P(x, y) = 0$ and $Q(x, y) = 0$. Classify a critical point (x_0, y_0) as stable or unstable node, stable or unstable spiral, center or saddle by computing the eigenvalues of the Jacobian matrix evaluated at the critical point (x_0, y_0):

$$J(x_0, y_0) = \begin{pmatrix} \dfrac{\partial P}{\partial x} & \dfrac{\partial P}{\partial y} \\ \dfrac{\partial Q}{\partial x} & \dfrac{\partial Q}{\partial y} \end{pmatrix}\Bigg|_{(x_0,y_0)}.$$

Two positive eigenvalues means an unstable node, two negative means a stable node, one positive and one negative means a saddle. Complex conjugate eigenvalues $\lambda = \alpha \pm i\beta$: $\alpha > 0$ means an unstable spiral, $\alpha < 0$ means a stable spiral.
Orthogonal Functions and Fourier Series: $\{\phi_0(x), \phi_1(x), \ldots\}$ orthogonal set on $[a, b]$ if $\int_a^b \phi_n(x)\phi_m(x)\,dx = 0$, $n \neq m$. The set of trigonometric functions form an orthogonal set on $[-L, L]$:

$$\left\{1, \cos\left(\frac{\pi x}{L}\right), \sin\left(\frac{\pi x}{L}\right), \cos\left(\frac{2\pi x}{L}\right), \sin\left(\frac{2\pi x}{L}\right), \ldots\right\}.$$

Fourier Series for $f(x)$ defined on $[-L, L]$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right].$$

The coefficients, a_0, a_n, b_n are computed from the orthogonality of the trigonometric functions on $[-L, L]$. The Fourier series simplify if the function $f(x)$ is even or odd. The Fourier cosine series (even function) and Fourier sine series (odd function):

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \quad \text{and} \quad f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right).$$

All Fourier series have period $2L$. At a jump discontinuity x_0 of a function $f(x)$, the Fourier series (or Fourier cosine or sine series) converges to $[f(x_0+) + f(x_0-)]/2$. Write a Fourier series, Fourier cosine series or Fourier sine series for a function $f(x)$ on $[-L, L]$ or $[0, L]$ and graph the periodic extensions.

Practice Problems:

1. The solution of $X'(t) = AX(t)$ is $X(t) = e^{2t}\left[c_1\begin{pmatrix}2\cos(t)\\ \sin(t)\end{pmatrix} + c_2\begin{pmatrix}-2\sin(t)\\ \cos(t)\end{pmatrix}\right]$. What are the eigenvalues of matrix A? Is the origin a stable or unstable node, a stable or unstable spiral or a saddle?

2. Compute the general solution of $X'(t) = A\vec{X}(t)$, where $A = \begin{pmatrix}2 & -1\\ 1 & 4\end{pmatrix}$. Write the unique solution for the initial condition $\vec{X}(0) = \begin{pmatrix}4\\ 1\end{pmatrix}$.

3. For the nonlinear autonomous system $dx/dt = x(5-y)$ and $dy/dt = y(x+y-4)$ compute all of the critical points and the Jacobian matrix. Classify each critical point as a stable or unstable node, stable or unstable spiral, or saddle.

4. Write the Fourier series for $f(x) = \begin{cases} 1, & -\pi < x < 0, \\ 3, & 0 < x < \pi. \end{cases}$ Graph the Fourier series for $-3\pi < x < 3\pi$. To what value does the Fourier series converge at $x = 0$? $x = \pi$? $x = 2\pi$?