Math 4354, Practice Problems, Exam 3

#3 $u(x,t) = v(x,t) + \psi(x)$

$v_t = v_{xx}$ $0<x<1$, $0<t<\infty$ $\quad \psi'' + 3x^2 - 1 = 0$

$v(0,t)=0$, $v(1,t)=0$, $0<t<\infty$ $\quad \psi(0)=2$, $\psi(1)=1$

$v(x,t) = \sum_{n=1}^{\infty} a_n e^{-(n\pi)^2 t} \sin(n\pi x)$ $\quad \boxed{\psi(x) = \frac{x^2}{2} - \frac{x^4}{4} - \frac{5}{4}x + 2}$ steady-state

$u(x,0) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) + \psi(x) = 2 \Rightarrow \sum_{n=1}^{\infty} a_n \sin(n\pi x) = -\frac{x^2}{2} + \frac{x^4}{4} + \frac{5}{4}x$

By Fourier sine series on $[0,1]$, $\boxed{a_n = 2\int_0^1 \left(-\frac{x^2}{2} + \frac{x^4}{4} + \frac{5}{4}x\right)\cdot \sin(n\pi x)\,dx}$

$\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n e^{-(n\pi)^2 t} \sin(n\pi x) + \psi(x)}$

#4 $u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos(n\theta) + B_n \sin(n\theta))$

$u(2,\theta) = A_0 + \sum_{n=1}^{\infty} 2^n (A_n \cos(n\theta) + B_n \sin(n\theta)) = 5 - 7\sin(4\theta)$

$A_0 = 5$ $\quad n=4$ $\quad 2^4 B_4 = -7$ $\quad \boxed{B_4 = \frac{-7}{2^4}}$ $\quad A_n = 0$ for all n $\quad B_n = 0$ for $n \neq 4$

$\boxed{u(r,\theta) = 5 - 7\left(\frac{r}{2}\right)^4 \sin(4\theta)}$

Max Temperature = 12
Min Temperature = -2

5.(a) Use Laplace transforms since homogeneous I.C.

$\mathcal{L}\{u_t\} = \mathcal{L}\{u_{xx}\}$ $\quad$ Assume $\lim_{x\to\infty} u(x,t) = 0$

$sU(x,s) - u(x,0) = \frac{d^2U}{dx^2} \Rightarrow \frac{d^2U}{dx^2} - sU = 0 \Rightarrow U(x,s) = c_1 e^{-\sqrt{s}x} + c_2 e^{\sqrt{s}x}$

Assume $\lim_{x\to\infty} u(x,t) = 0$ and $\lim_{x\to\infty} U(x,t) = 0$ $\quad \mathcal{L}\{u(0,t)\} = \mathcal{L}\{2\} = \frac{2}{s}$, $U(0,s) = \frac{2}{s}$

$\Rightarrow U(x,s) = \frac{2}{s} e^{-\sqrt{s}x} \Rightarrow \mathcal{L}^{-1}\{U(x,s)\} = \mathcal{L}^{-1}\left\{\frac{2}{s} e^{-\sqrt{s}x}\right\} = 2\mathcal{L}^{-1}\left\{\frac{e^{-\sqrt{s}x}}{s}\right\} = 2\operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right)$

Therefore, $\boxed{u(x,t) = 2\operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right)}$

(b) $u(x,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} e^{-k((n\pi)^2 + (m\pi)^2)t} \sin(n\pi x)\sin(n\pi y)$

$u(x,y,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} \sin(n\pi x)\sin(m\pi y) = \sin(2\pi x)\sin(3\pi y) + 9\sin(\pi x)\sin(5\pi y)$

$u(x,y,t) = e^{-13k\pi^2 t}\sin(2\pi x)\sin(3\pi y) + 9e^{-26\pi^2 k t}\sin(\pi x)\sin(5\pi y)$

(c) $\boxed{u(x,t) = 4e^{-4t} + 2e^{-(4+9k)t}\cos(3x)}$

(d) $\mathcal{F}_c\{u_{tt}\} = \mathcal{F}_c\{u_{xx}\}$ $\quad 0$

$\frac{d^2U}{dt^2} = -\alpha^2 U(\alpha,t) - u_x(0,t) \Rightarrow \frac{d^2U}{dt^2} + \alpha^2 U = 0 \Rightarrow U = c_1\cos(\alpha t) + c_2 \sin(\alpha t)$

$\mathcal{F}_c\{u(x,0)\} = \mathcal{F}_c\{4e^{-x}\}$ and $\frac{dU}{dt}(\alpha,0) = 0 \Rightarrow c_1 = \frac{4}{\alpha^2+1}$, $c_2 = 0$

$U(\alpha,0) = 4/(\alpha^2+1)$

Take inverse transforms $\mathcal{F}_c^{-1}\{U(\alpha,t)\} = \mathcal{F}_c^{-1}\left\{\frac{4}{\alpha^2+1}\cos(\alpha t)\right\}$

Therefore $u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{4}{\alpha^2+1}\cos(\alpha t)\cos(\alpha x)\,d\alpha$