3 Practice Problems:

1. Compute the general solution of $d\vec{X}(t)/dt = A\vec{X}(t)$, where $A = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}$. Write the unique solution for the initial condition $\vec{X}(0) = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

2. For the nonlinear autonomous system $dx/dt = x(5-y)$ and $dy/dt = y(x+y-4)$ compute all of the critical points and use the Jacobian matrix to classify each critical point as a node, saddle, or spiral and stable or unstable.

3. Write the Fourier sine series for $f(x) = 3$ for $0 < x < 1$ and evaluate the coefficients b_n. Write out the series up to $n = 5$ terms. The series has period 2. Graph the series on $-3 < x < 3$.

Answers

1: $\lambda^2 - 6\lambda + 9 = 0$ $(\lambda-3)^2 = 0$ $\lambda = 3, 3$ $\begin{pmatrix} 2-3 & -1 \\ 1 & 4-3 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $k_1 + k_2 = 0$, $k_2 = -k_1$ $\vec{K} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ Repeated roots only one eigenvector

$(A-3I)\vec{P} = \vec{K} \Rightarrow \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow -P_1 - P_2 = 1$ or $P_1 + P_2 = -1$ $\Rightarrow P_2 = -1, P_1 = 0$ $\vec{P} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ one solution

$$\boxed{\vec{X}(t) = c_1 e^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{3t}\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right]}$$

$\vec{X}(0) = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ $\left.\begin{matrix} c_1 = 4 \\ -c_1 - c_2 = 1 \end{matrix}\right\} \Rightarrow \boxed{\begin{matrix} c_1 = 4 \\ c_2 = -5 \end{matrix}}$

$$\boxed{\vec{X}(t) = 4e^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} - 5e^{3t}\begin{pmatrix} t \\ -t-1 \end{pmatrix}}$$

2. Critical Points: $(0,0)$, $(0,4)$, $(-1,5)$ $J(x,y) = \begin{pmatrix} 5-y & -x \\ y & x+2y-4 \end{pmatrix}$

$J(0,0) = \begin{pmatrix} 5 & 0 \\ 0 & -4 \end{pmatrix}$ $\lambda = 5, -4$ $(0,0)$ is a saddle unstable

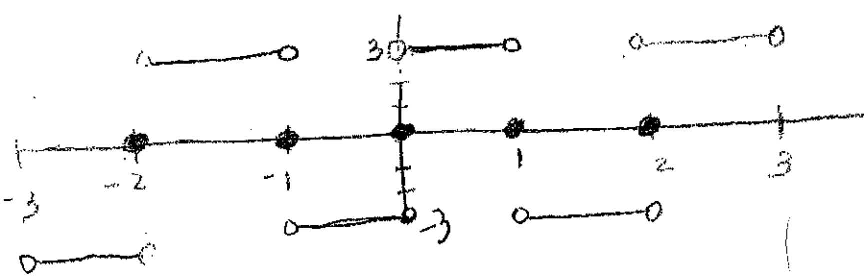
$J(0,4) = \begin{pmatrix} 1 & 0 \\ 4 & 4 \end{pmatrix}$ $\lambda = 1, 4$ $(0,4)$ is an unstable node

$J(-1,5) = \begin{pmatrix} 0 & 1 \\ 5 & 5 \end{pmatrix}$ $\lambda^2 - 5\lambda - 5 = 0$ $\lambda = \frac{5 \pm \sqrt{45}}{2}$ $(-1,5)$ unstable saddle

3. $L = 1$

$$b_n = \frac{2}{1}\int_0^1 3\sin(n\pi x)\,dx = -6\frac{\cos(n\pi x)}{n\pi}\Big|_0^1 = -6\frac{\overbrace{\cos(n\pi)}^{(-1)^n}}{n\pi} + \frac{6}{n\pi} = \frac{6}{n\pi}\left(1 - (-1)^n\right)$$

$$f(x) = \sum_{n=1}^{\infty} \frac{6}{n\pi}\left(1 - (-1)^n\right)\sin(n\pi x) = \underset{1}{\frac{12}{\pi}\sin(\pi x)} + \underset{2}{0} + \underset{3}{\frac{12}{3\pi}\sin(3\pi x)} + \underset{4}{0} + \underset{5}{\frac{12}{5\pi}\sin(5\pi x)} + \dots$$



odd

Practice Problems:

1. The solution of $X'(t) = AX(t)$ is $X(t) = e^{2t}\left[c_1\begin{pmatrix}2\cos(t)\\ \sin(t)\end{pmatrix} + c_2\begin{pmatrix}-2\sin(t)\\ \cos(t)\end{pmatrix}\right]$. What are the eigenvalues of matrix A? Is the origin a stable or unstable node, a stable or unstable spiral or a saddle?

2. Compute the general solution of $X'(t) = A\vec{X}(t)$, where $A = \begin{pmatrix}2 & -1\\ 1 & 4\end{pmatrix}$. Write the unique solution for the initial condition $\vec{X}(0) = \begin{pmatrix}4\\ 1\end{pmatrix}$.

3. For the nonlinear autonomous system $dx/dt = x(5-y)$ and $dy/dt = y(x+y-4)$ compute all of the critical points and the Jacobian matrix. Classify each critical point as a stable or unstable node, stable or unstable spiral, or saddle.

4. Write the Fourier series for $f(x) = \begin{cases} 1, & -\pi < x < 0, \\ 3, & 0 < x < \pi. \end{cases}$ Graph the Fourier series for $-3\pi < x < 3\pi$. To what value does the Fourier series converge at $x = 0$? $x = \pi$? $x = 2\pi$?

Answers:

① $\lambda_{1,2} = 2 \pm i$ $\quad \alpha = 2,\ \beta = 1,$ Since $\alpha > 0$, the origin is an unstable spiral

④ $L = \pi$ $\quad a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} 1\,dx + \int_{0}^{\pi} 3\,dx\right] = \frac{1}{\pi}\left[+\pi + 3\pi\right] = 4 \Rightarrow \frac{a_0}{2} = 2$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} \cos(nx)\,dx + \int_{0}^{\pi} 3\cos(nx)\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin(nx)}{n}\Big|_{-\pi}^{0}\right] + \frac{3}{\pi}\left[\frac{\sin(nx)}{n}\Big|_{0}^{\pi}\right]$$

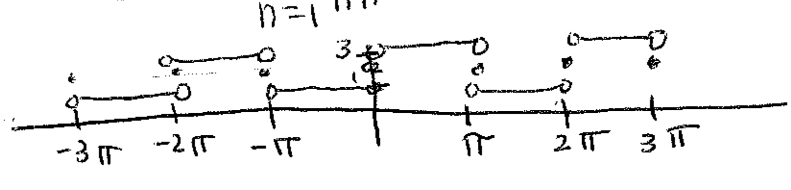
$$= \frac{1}{n\pi}\left[\underbrace{\sin(0)}_{0} - \underbrace{\sin(-n\pi)}_{0}\right] + \frac{3}{n\pi}\left[\underbrace{\sin(n\pi)}_{0} - \underbrace{\sin(0)}_{0}\right] = 0 \qquad a_n = 0$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} \sin(nx)\,dx + \int_{0}^{\pi} 3\sin(nx)\,dx\right] = \frac{1}{\pi}\left[-\frac{\cos(nx)}{n}\Big|_{-\pi}^{0}\right] + \frac{3}{\pi}\left[-\frac{\cos(nx)}{n}\Big|_{0}^{\pi}\right]$$

$$= \frac{1}{n\pi}\left[\underbrace{-\cos(0)}_{-1} + \underbrace{\cos(-n\pi)}_{\cos(n\pi)}\right] + \frac{3}{n\pi}\left[-\cos(n\pi) + \underbrace{\cos(0)}_{+1}\right] \quad \text{combine like terms}$$

$$= \frac{2}{n\pi}\left[1 - \underbrace{\cos(n\pi)}_{(-1)^n}\right] = \frac{2}{n\pi}\left[1 - (-1)^n\right] = b_n$$

$$f(x) = 2 + \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1 - (-1)^n\right]\sin(nx)$$



At $x = 0, \pi, 2\pi$, F.S. converges to $\frac{3+1}{2} = 2$