Laplace Transforms to Solve BVPs for PDEs

Laplace transforms can be used solve linear PDEs. Laplace transforms applied to the t variable (change to s) and the PDE simplifies to an ODE in the x variable. Recall the Laplace transform for $f(t)$.

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s), \quad \mathcal{L}^{-1}\{F(s)\} = f(t)$$

Apply the Laplace transform to $u(x,t)$ and to the PDE.

$$\boxed{\mathcal{L}\{u(x,t)\} = U(x,s), \quad \mathcal{L}^{-1}\{U(x,s)\} = u(x,t)}$$

The Laplace transform changes the derivatives with respect to t but NOT x:

$$\begin{aligned}
\mathcal{L}\{u_t\} &= s\mathcal{L}\{u(x,t)\} - u(x,0) = sU(x,s) - u(x,0) \\
\mathcal{L}\{u_{tt}\} &= s^2 U(x,s) - su(x,0) - u_t(x,0) \\
\mathcal{L}\{u_{xx}\} &= \int_0^\infty e^{-st} \frac{\partial^2 u}{\partial x^2}\,dt = \frac{d^2}{dx^2}\int_0^\infty e^{-st} u(x,t)\,dt = \frac{d^2 U}{dx^2}.
\end{aligned}$$

Apply the Laplace transform to the heat equation, $ku_{xx} = u_t$ and the wave equation $a^2 u_{xx} = u_{tt}$:

Heat: $k\mathcal{L}\{u_{xx}\} = \mathcal{L}\{u_t\} \quad \Rightarrow \quad k\dfrac{d^2U}{dx^2} = sU - u(x,0)$

Wave: $a^2\mathcal{L}\{u_{xx}\} = \mathcal{L}\{u_{tt}\} \quad \Rightarrow \quad a^2\dfrac{d^2U}{dx^2} = s^2U - su(x,0) - u_t(x,0)$

Example 1: Solve the heat equation with Laplace transforms: $\begin{cases} u_{xx} = u_t, \ 0 < x < \infty, \quad 0 < t < \infty \\ u(0,t) = 1, \ \lim\limits_{x\to\infty} u(x,t) = 2, \quad 0 < t < \infty \\ u(x,0) = 2, \ 0 < x < \infty \end{cases}$

The Laplace transform changes the PDE in x and t to an ODE in x: $\mathcal{L}\{u_{xx}\} = \mathcal{L}\{u_t\}$

$$\begin{aligned}
\frac{d^2U}{dx^2} &= sU - u(x,0) \\
\frac{d^2U}{dx^2} - sU &= -2
\end{aligned}$$

Solve the ODE. $U = U_c + U_p$, $U_c = c_1 e^{-\sqrt{s}x} + c_2 e^{\sqrt{s}x}$ and $U_p = A$. Solving for A, $U_p = \dfrac{2}{s}$.

$$U(x,s) = c_1 e^{-\sqrt{s}x} + c_2 e^{\sqrt{s}x} + \frac{2}{s}$$

Apply the Laplace transform to BC, $x = 0$, $x \to \infty$: $\mathcal{L}\{u(0,t)\} = \mathcal{L}\{1\}$ and assume $\lim\limits_{x\to\infty} \mathcal{L}\{u(x,t)\} = \mathcal{L}\{2\}$

$$U(0,s) = \frac{1}{s} \quad \text{and} \quad \lim_{x\to\infty} U(x,s) = \frac{2}{s}$$

Solve for c_1 and c_2 : $U(0,s) = \dfrac{1}{s} = c_1 + c_2 + \dfrac{2}{s}$. As $x \to \infty$, $c_2 = 0$. So $c_1 = -\dfrac{1}{s}$

$$\boxed{U(x,s) = -\frac{e^{-\sqrt{s}x}}{s} + \frac{2}{s}}$$

Take the inverse Laplace transform and use the table.

$$\mathcal{L}^{-1}\{U(x,s)\} = -\mathcal{L}^{-1}\left\{\frac{e^{-\sqrt{s}x}}{s}\right\} + 2\mathcal{L}\left\{\frac{1}{s}\right\}$$

$$\boxed{u(x,t) = -\operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right) + 2}$$

Example 2: Solve the wave equation with Laplace transforms: $\begin{cases} u_{xx} = u_{tt}, \ 0 < x < 1, \ 0 < t < \infty \\ u(0,t) = 0, \ u(1,t) = 0, \ 0 < t < \infty \\ u(x,0) = 0, \ u_t(x,0) = \sin(\pi x), \ 0 < x < 1. \end{cases}$

The Laplace transform changes the PDE to an ODE in x, $\mathcal{L}\{u_{xx}\} = \mathcal{L}\{u_{tt}\}$.

$$\begin{aligned} \frac{d^2U}{dx^2} &= s^2U - su(x,0) - u_t(x,0) \\ \frac{d^2U}{dx^2} - s^2U &= -\sin(\pi x) \end{aligned}$$

Solve the ODE: $U = U_c + U_p$: $U_c = c_1e^{-sx} + c_2e^{sx}$ and $U_p = A\sin(\pi x) + B\cos(\pi x)$. Solving for A and B by substituting into the ODE,

$$-A\pi^2\sin(\pi x) - B\pi^2\cos(\pi x) - s^2(A\sin(\pi x) + B\cos(\pi x)) = -\sin(\pi x)$$

and matching coefficients: $-A\pi^2 - As^2 = -1, \ -B\pi^2 - Bs^2 = 0$ implies

$$A = \frac{1}{\pi^2 + s^2} \quad \text{and} \quad B = 0, \quad \text{so that } U_p = \frac{1}{\pi^2 + s^2}\sin(\pi x)$$

The solution is

$$U(x,s) = c_1e^{-sx} + c_2e^{sx} + \frac{1}{\pi^2 + s^2}\sin(\pi x)$$

Apply the BC to find c_1 and c_2, $U(0,s) = 0$, $U(1,s) = 0$: $c_1 = 0$ and $c_2 = 0$.

$$\boxed{U(x,s) = \frac{1}{\pi^2 + s^2}\sin(\pi x)}$$

Take the inverse Laplace transform and use the table:

$$\mathcal{L}^{-1}\{U(x,s\} = \mathcal{L}^{-1}\left\{\frac{1}{\pi^2 + s^2}\right\}\underbrace{\sin(\pi x)}_{\text{No } s \text{ variable}}$$

$$\boxed{u(x,t) = \frac{1}{\pi}\sin(\pi t)\sin(\pi x).}$$

Also, the BVP can be solved using separation of variables and the homogeneous BC:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty}[a_n\cos(n\pi t) + b_n\sin(n\pi t)]\sin(n\pi x)}$$

Applying the IC $u(x,0) = 0$ and $u_t(x,0) = \sin(\pi x)$ leads to

$$\begin{aligned} u(x,0) &= 0 = \sum_{n=1}^{\infty} a_n\sin(n\pi x) \Longrightarrow a_n = 0, \ n = 1, 2, \ldots. \\ u_t(x,0) &= \sin(\pi x) = \sum_{n=1}^{\infty} b_n n\pi\sin(n\pi x) \Longrightarrow b_1\pi = 1, \ b_n = 0, \ n = 2, 3, \ldots. \end{aligned}$$

The same solution is obtained: $\boxed{u(x,t) = \frac{1}{\pi}\sin(\pi t)\sin(\pi x)}$.

Homework Problems

1. Solve the BVP for the heat equation 14.2 # 11.
2. Solve the BVP for the wave equation using two methods: (1) Laplace transforms and (2) Separation of Variables. Show the solution is $u(x,t) = 4\cos(3\pi t)\sin(3\pi x) + \frac{2}{\pi}\sin(\pi t)\sin(\pi x)$.

   $$\begin{cases} u_{xx} = u_{tt}, \ 0 < x < 1, \ 0 < t < \infty \\ u(0,t) = 0, \ u(1,t) = 0, \ 0 < t < \infty \\ u(x,0) = 4\sin(3\pi x), \quad u_t(x,0) = 2\sin(\pi x), \ 0 < x < 1. \end{cases}$$