Heat, Wave, Laplace's Equation

Separation of Variables: $u(x,t) = X(x)T(t)$, $u(x,y) = X(x)Y(y)$

BC: $u(0,t) = 0$, $u(L,t) = 0$ lead to a eigenvalue problem:
$X'' + \lambda X = 0$, $X(0) = 0$, $X(L) = 0$, $X_n(x) = \sin(n\pi x/L)$, $\lambda_n = (n\pi/L)^2$, $n = 1, 2, \ldots$.

BC: $u_x(0,t) = 0$, $u_x(L,t) = 0$ lead to an eigenvalue problem:
$X'' + \lambda X = 0$, $X'(0) = 0$, $X'(L) = 0$, $X_n(x) = \cos(n\pi x/L)$, $\lambda_n = (n\pi/L)^2$, $n = 0, 1, 2, \ldots$.

Heat Equation (Parabolic): $u_t = ku_{xx}$, $0 < x < L$, $t > 0$, $X'' + \lambda X = 0$, $T' + k\lambda T = 0$.

$$\boxed{u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-\lambda_n kt} X_n(x)}$$

1. If the homogeneous boundary conditions are $u(0,t) = 0$ and $u(L,t) = 0$, then
$A_0 =$ 0, $\lambda_n =$ $\left(\frac{n\pi}{L}\right)^2$, $X_n(x) =$ $\sin\left(\frac{n\pi}{L}x\right)$.
2. If the homogeneous boundary conditions are $u_x(0,t) = 0$ and $u_x(L,t) = 0$, then
$\lambda_n =$ $\left(\frac{n\pi}{L}\right)^2$, $X_n(x) =$ $\cos\left(\frac{n\pi}{L}x\right)$.

Wave equation (Hyperbolic): $u_{tt} = a^2 u_{xx}$, $0 < x < L$, $t > 0$, $X'' + \lambda X = 0$, $T'' + a\lambda T = 0$

$$\boxed{u(x,t) = A_0 + B_0 t + \sum_{n=1}^{\infty} [A_n \cos(a\alpha_n t) + B_n \sin(a\alpha_n t)] X_n(x)}$$

3. If the homogeneous boundary conditions are $u(0,t) = 0$ and $u(L,t) = 0$, then
$A_0 =$ 0, $B_0 =$ 0, $\alpha_n =$ $\frac{n\pi}{L}$, $X_n(x) =$ $\sin\left(\frac{n\pi}{L}x\right)$.
4. If the homogeneous boundary conditions are $u_x(0,t) = 0$ and $u_x(L,t) = 0$, then
$\alpha_n =$ $\frac{n\pi}{L}$, $X_n(x) =$ $\cos\left(\frac{n\pi}{L}x\right)$.

Laplace's equation (Elliptic): $u_{xx} + u_{yy} = 0$, $0 < x < a$, $0 < y < b$.

BC homogeneous in X: $X'' + \lambda X = 0$, $Y'' - \lambda Y = 0$:

$$\boxed{u(x,y) = A_0 + B_0 y + \sum_{n=1}^{\infty} [A_n \cosh(\alpha_n y) + B_n \sinh(\alpha_n y)] X_n(x)}$$

5. If the homogeneous boundary conditions are $u(0,y) = 0$ and $u(a,y) = 0$, then
$A_0 =$ 0, $B_0 =$ 0, $\alpha_n =$ $\frac{n\pi}{a}$, $X_n(x) =$ $\sin\left(\frac{n\pi x}{a}\right)$.
6. If the homogeneous boundary conditions are $u_x(0,y) = 0$ and $u_x(a,y) = 0$, then
$\alpha_n =$ $\frac{n\pi}{a}$, $X_n(x) =$ $\cos\left(\frac{n\pi x}{a}\right)$.

BC homogeneous in Y: $Y'' + \lambda Y = 0$, $X'' - \lambda X = 0$.

$$\boxed{u(x,y) = A_0 + B_0 x + \sum_{n=1}^{\infty} [A_n \cosh(\alpha_n x) + B_n \sinh(\alpha_n x)] Y_n(y)}$$

7. If the homogeneous boundary conditions are $u(x,0) = 0$ and $u(x,b) = 0$, then
$A_0 =$ 0, $B_0 =$ 0, $\alpha_n =$ $\frac{n\pi}{b}$, $Y_n(y) =$ $\sin\left(\frac{n\pi}{b}y\right)$
8. If the homogeneous boundary conditions are $u_y(x,0) = 0$ and $u_y(x,b) = 0$, then
$\alpha_n =$ $\frac{n\pi}{b}$, $Y_n(y) =$ $\cos\left(\frac{n\pi}{b}y\right)$

1. Solve the BVP for the heat equation.

$u_t = ku_{xx}$, $0 < x < 2$, $t > 0$
$u_x(0,t) = 0$, $u_x(2,t) = 0$, $t > 0$
$u(x,0) = f(x)$, $0 < x < 2$

$X'' + \lambda X = 0$ $\quad T' + \lambda k T = 0$
$X'(0) = 0$ $\quad T_n = a_n e^{-(\frac{n\pi}{2})^2 kt}$
$X'(2) = 0$
$\lambda_n = (\frac{n\pi}{2})^2$ $\quad X_n = \cos(\frac{n\pi x}{2})$ $\quad n = 0,1,2$

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-(\frac{n\pi}{2})^2 kt} \cos(\frac{n\pi x}{2})$$

$$u(x,0) = f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi x}{2}) \Rightarrow$$ use Fourier Cosine Series But the first coefficient is $\frac{c_0}{2}$ *

$$\boxed{C_0 = \frac{2}{2}\int_0^2 f(x)\,dx,\ C_n = \frac{2}{2}\int_0^2 f(x)\cos(\frac{n\pi x}{2})\,dx}$$

Answer: $$u(x,t) = \frac{\int_0^2 f(x)\,dx}{2} + \sum_{n=1}^{\infty} \int_0^2 f(x)\cos(\frac{n\pi}{2}x)\,dx\, e^{-kt(\frac{n\pi}{2})^2} \cos(\frac{n\pi x}{2})$$

2. Solve the BVP for the wave equation.

$u_{tt} = u_{xx}$, $0 < x < 2$, $t > 0$
$u(0,t) = 0$, $u(2,t) = 0$, $t > 0$
$u(x,0) = f(x)$, $u_t(x,0) = 0$, $0 < x < 2$

$X'' + \lambda X = 0$ $\quad T'' + \lambda T = 0$
$X(0) = 0$ $\quad T_n = A_n\cos(\frac{n\pi}{2}t) + B_n\sin(\frac{n\pi}{2}t)$
$X(2) = 0$
$X_n = \sin(\frac{n\pi x}{2})$ $\quad n = 1,2,$
$\lambda_n = (\frac{n\pi}{2})^2$

$$u(x,t) = \sum_{n=1}^{\infty}[A_n\cos(\frac{n\pi}{2}t) + B_n\sin(\frac{n\pi}{2}t)]\sin(\frac{n\pi x}{2})$$

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n\sin(\frac{n\pi x}{2}) \qquad \boxed{A_n = \frac{2}{2}\int_0^2 f(x)\sin(\frac{n\pi x}{2})\,dx}$$

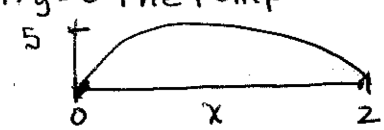
$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} \frac{n\pi}{2}B_n \sin(\frac{n\pi x}{2}) = 0 \Rightarrow \boxed{B_n = 0}$$

Answer: $$u(x,t) = \sum_{n=1}^{\infty}\int_0^2 f(x)\sin(\frac{n\pi x}{2})\,dx\,\cos(\frac{n\pi}{2}t)\cdot\sin(\frac{n\pi x}{2})$$

3. Solve the BVP for the steady-state temperature on the rectangle.

$u_{xx} + u_{yy} = 0$, $0 < x < 2$, $0 < y < 4$
$u(0,y) = 0$, $u(2,y) = 0$, $0 < y < 4$
$u(x,0) = 5\sin(\pi x/2)$, $u(x,4) = 0$, $0 < x < 2$

On $y=0$ the temperature is $5\sin(\frac{\pi x}{2})$



On $x=0$, $y=4$ and $x=2$, the temperature is Zero. Max and Min of Laplace's Eqn occurs on Boundary.

$X'' + \lambda X = 0$ $\quad Y'' - \lambda Y = 0 \Rightarrow \cosh, \sinh$
$X(0) = 0$ $\quad X_n = \sin(\frac{n\pi x}{2})$ $\quad n = 1,2,3,\ldots$
$X(2) = 0$ $\quad \lambda_n = (\frac{n\pi}{2})^2$

$$u(x,y) = \sum_{n=1}^{\infty}[A_n\cosh(\frac{n\pi y}{2}) + B_n\sinh(\frac{n\pi y}{2})]\sin(\frac{n\pi x}{2})$$ $A_1 = 5$, $A_n = 0, n \neq 1$

$$u(x,0) = \sum_{n=1}^{\infty} A_n\sin(\frac{n\pi x}{2}) = 5\sin(\frac{\pi x}{2})$$

$$u(x,4) = \sum_{n=1}^{\infty}[A_n\cosh(2n\pi) + B_n\sinh(2n\pi)]\sin(\frac{n\pi x}{2}) = 0 \Rightarrow B_1 = -\frac{5\cosh(2\pi)}{\sinh(2\pi)}$$ at $n=1$; $B_n = 0$, $n \neq 1$

Answer: $$u(x,y) = [5\cosh(\frac{\pi y}{2}) - \frac{5\cosh(2\pi)}{\sinh(2\pi)}\sinh(\frac{\pi y}{2})]\sin(\frac{\pi x}{2})$$

What is the maximum temperature on the rectangle? 5

What is the minimum temperature on the rectangle? 0