Two Eigenvalue Problems and Two Generalized Fourier Series

(1) $\begin{cases} X'' + \lambda X = 0, \quad 0 < x < L \\ X(0) = 0 \\ X'(L) = 0 \end{cases}$ (2) $\begin{cases} X'' + \lambda X = 0, \quad 0 < x < L \\ X'(0) = 0 \\ X(L) = 0 \end{cases}$

For each problem, we check for nonzero solutions for X by letting $\lambda = -\alpha^2$, $\lambda = 0$ and $\lambda = +\alpha^2$. The only case that gives nonzero solutions is $\lambda = +\alpha^2$.

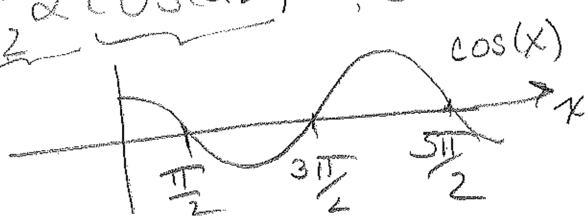
(1) $\lambda = \alpha^2$ $\begin{cases} X = c_1 \cos(\alpha x) + c_2 \sin(\alpha x) \\ X' = c_1 \alpha(-\sin(\alpha x)) + c_2 \alpha \cos(\alpha x) \end{cases}$

Now apply the boundary conditions.

$0 = X(0) = c_1 \cos(0) + c_2 \sin(0) = \underline{c_1 = 0}$

Thus, $\underline{X = c_2 \sin(\alpha x)}$ $\quad c_2 \neq 0, \alpha \neq 0$

$0 = X'(L) = \cancel{c_1 \alpha(-\sin(\alpha L))} + c_2 \alpha \cos(\alpha L) \stackrel{\text{set}}{=} 0$



Therefore $\underline{\cos(\alpha L) = 0}$

$\alpha L = \frac{2n-1}{2}\pi$

$\alpha_n = \frac{2n-1}{2L}\pi \Rightarrow \boxed{\lambda_n = \left(\frac{2n-1}{2L}\pi\right)^2}$

In case 1 $\boxed{\begin{aligned} X_n &= \sin\left(\frac{2n-1}{2L}\pi x\right) \\ \lambda_n &= \left(\frac{2n-1}{2L}\pi\right)^2 \end{aligned}}$ $n=1,2,\ldots$

We do the same procedure for case 2 but use boundary conditions $\underline{X'(0)=0}$ and $\underline{X(L)=0}$

$$0 = X'(0) = c_1\alpha(-\sin(0)) + c_2\alpha\cos(0) = c_2\alpha = 0$$

$$\underline{c_2 = 0}$$

$$0 = X(L) = \underbrace{c_1}\underbrace{\cos(\alpha L)} + \cancel{c_2\sin(\alpha L)} = 0$$

$$\cos(\alpha L) = 0$$

$$\alpha L = \frac{2n-1}{2}\pi \qquad \alpha_n = \frac{2n-1}{2L}\pi.$$

Therefore, in case 2 $\boxed{\begin{aligned} X_n(x) &= \cos\left(\frac{2n-1}{2L}\pi x\right) \\ \lambda_n &= \left(\frac{2n-1}{2L}\pi\right)^2, \quad n=1,2,\ldots \end{aligned}}$

Generalized Fourier series

(1)
$$\begin{cases} f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{2n-1}{2L}\pi x\right) \\ A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{2n-1}{2L}\pi x\right)dx \end{cases}$$

Why? $\int_0^L \sin^2\left(\frac{2n-1}{2L}\pi x\right)dx = \frac{L}{2}$

(2)
$$\begin{cases} f(x) = \sum_{n=1}^{\infty} B_n \cos\left(\frac{2n-1}{2L}\pi x\right) \\ B_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{2n-1}{2L}\pi x\right)dx \end{cases}$$