Exam #2A Math 4354
Fall 2018

Name: Answers

Instructions: Turn off all cell phones and electronic devices. No calculators are allowed. Please write neatly and clearly. Put answers in spaces provided. Show all of your work.

1. (12 pts) Compute the eigenvalues λ_n and eigenfunctions $X_n(x)$.

   $X'' + \lambda X = 0$, $X(0) = 0$, $X(10) = 0$?

   $\lambda_n =$ $\left(\frac{n\pi}{10}\right)^2$ $X_n(x) =$ $\sin\left(\frac{n\pi x}{10}\right)$, $n =$ $1, 2, \ldots$

   $X'' + \lambda X = 0$, $X'(0) = 0$, $X'(\pi) = 0$?

   $\lambda_n =$ $(n)^2$ $X_n(x) =$ $\cos(nx)$, $n =$ $0, 1, 2, \ldots$

2. (6 pts) Determine whether the PDE is hyperbolic, parabolic, or elliptic.

   (a) $u_{xx} - 5u_{xy} + u_y + u = 0$ hyperbolic $\quad B^2 - 4AC = (5)^2 - 4(1)(0) > 0$

   (b) $u_t + 16u_{xx} + u = 0$ parabolic $\quad B^2 - 4AC = (0)^2 - 4(16)(0) = 0$

3. (6 pts) Separate variables, $u(x,t) = X(x)T(t)$, for the following PDE: $u_t - 4u_x = u_{xx}$

$$\frac{XT'}{XT} - \frac{4X'T}{XT} = \frac{X''T}{XT} \Rightarrow \frac{T'}{T} - \frac{4X'}{X} = \frac{X''}{X}$$

$$\boxed{\frac{T'}{T} = \frac{X''}{X} + \frac{4X'}{X}}$$

4. (6 pts) Solve the BVP for the heat equation.
   $u_t = u_{xx}$, $0 < x < 1$, $t > 0$
   $u_x(0,t) = 0$, $u_x(1,t) = 0$, $t > 0$
   $u(x,0) = 5$, $0 < x < 1$.

$$u(x,t) = c_0 + \sum_{n=1}^{\infty} c_n e^{-(n\pi)^2 t} \cos(n\pi x)$$

$$u(x,0) = c_0 + \sum_{n=1}^{\infty} c_n \cos(n\pi x) = 5 \Rightarrow c_0 = 5,\ c_n = 0,\ n = 1, 2, \ldots$$

$$\boxed{u(x,t) = 5}$$

5. (20 pts) Solve the BVP for the heat equation.

$u_t = ku_{xx},\ 0 < x < 2,\ t > 0$

$u(0,t) = 0,\ u(2,t) = 0,\ t > 0.$

$$u(x,0) = \begin{cases} 3, & 0 < x < 1 \\ 0, & 1 < x < 2. \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\left(\frac{n\pi}{2}\right)^2 kt} \sin\left(\frac{n\pi x}{2}\right)$$

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{2}\right) = \begin{cases} 3, & 0 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$$

Apply Fourier sine coefficients.

$$C_n = \frac{2}{2}\int_0^1 3 \sin\left(\frac{n\pi x}{2}\right) dx + \frac{2}{2}\int_1^2 0 \cdot \sin\left(\frac{n\pi x}{2}\right) dx$$

$$= -\frac{3 \cdot 2}{n\pi} \cos\left(\frac{n\pi x}{2}\right)\Big|_0^1 = \frac{-6}{n\pi}\left(\cos\left(\frac{n\pi}{2}\right) - \cos(0)\right)$$

$$= \frac{6}{n\pi}\left(1 - \cos\left(\frac{n\pi}{2}\right)\right)$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{6}{n\pi}\left(1 - \cos\left(\frac{n\pi}{2}\right)\right) e^{-\left(\frac{n\pi}{2}\right)^2 kt} \sin\left(\frac{n\pi x}{2}\right)}$$

6. (20 pts) Solve the BVP for the wave equation.

$u_{tt} = u_{xx}$, $0 < x < 3$, $t > 0$

$u(0,t) = 0$, $u(3,t) = 0$, $t > 0$

$u(x,0) = 4\sin(2\pi x)$, $u_t(x,0) = \sin(3\pi x)$, $0 < x < 3$.

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{n\pi}{3}t\right) + B_n \cdot \sin\left(\frac{n\pi}{3}t\right)\right]\sin\left(\frac{n\pi x}{3}\right)$$

① $u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{3}\right) = 4\sin(2\pi x)$

② $u_t(x,0) = \sum_{n=1}^{\infty} \frac{n\pi}{3} B_n \sin\left(\frac{n\pi x}{3}\right) = \sin(3\pi x)$

① $\Rightarrow$ $A_6 = 4$, $A_n = 0$ $n \neq 6$

② $\Rightarrow$ $3\pi B_9 = 1$, $B_n = 0$, $n \neq 9$

$B_9 = \frac{1}{3\pi}$

Answer

$$\boxed{4\cos(2\pi t)\sin(2\pi x) + \frac{1}{3\pi}\sin(3\pi t)\sin(3\pi x)}$$

7. (10 pts) The solution of Laplace's equation on the rectangle $0 < x < 3$, $0 < y < 2$ is

$$u(x,y) = 4\left[\cosh(\pi x) - \frac{\cosh(3\pi)}{\sinh(3\pi)}\sinh(\pi x)\right]\sin(\pi y) + 5\frac{\sinh(2\pi y)}{\sinh(4\pi)}\sin(2\pi x).$$

What are the four boundary conditions at $x = 0$? $x = 3$? $y = 0$? $y = 2$?

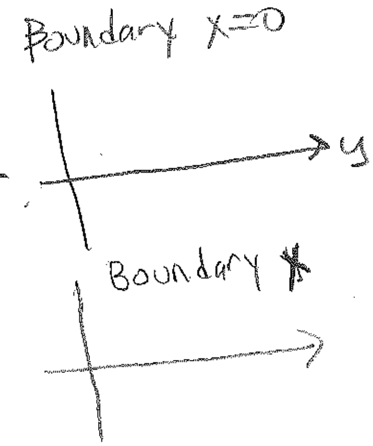
$u(0,y) = 4\sin(\pi y)$

$u(3,y) = 0$

$u(x,0) = 0$

$u(x,2) = 5\sin(2\pi x)$

Exam #2A Answers



8. (20 pts) Solve the BVP for Laplace's equation.
$u_{xx}+u_{yy}=0,\ 0<x<2,\ 0<y<2$
$u(0,y)=3\sin(\pi y),\ u(2,y)=\sin(4\pi y),\ 0<y<2$
$u(x,0)=0,\ u(x,2)=0,\ 0<x<2.$ Homogeneous BC on $y=0, y=2$

What is the maximum temperature on the rectangle? -3

What is the minimum temperature on the rectangle? 3

$X''-\lambda X=0 \qquad Y''+\lambda Y=0 \qquad \lambda_n=\left(\frac{n\pi}{2}\right)^2$

$Y(0)=0 \qquad Y_n=\sin\left(\frac{n\pi y}{2}\right)$

$X=A_n\cosh\left(\frac{n\pi x}{2}\right)+B_n\sinh\left(\frac{n\pi y}{2}\right) \qquad Y(2)=0$

$$\boxed{u(x,y)=\sum_{n=1}^{\infty}\left[A_n\cosh\left(\frac{n\pi x}{2}\right)+B_n\sinh\left(\frac{n\pi x}{2}\right)\right]\sin\left(\frac{n\pi y}{2}\right)}$$

Apply the BC at $x=0$ and $x=2$, $\cosh(0)=1, \sinh(0)=0$

$$u(0,y)=\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi y}{2}\right)=3\sin(\pi y)\Rightarrow \boxed{A_2=3,\ A_n=0,\ n\neq 2}$$

$$u(2,y)=\sum_{n=1}^{\infty}\left[A_n\cosh(n\pi)+B_n\sinh(n\pi)\right]\sin\left(\frac{n\pi y}{2}\right)=\sin(4\pi y)$$

Because the BC have $\sin(\pi y)$ and $\sin(4\pi y)$ there are only 2 $\sin\left(\frac{n\pi y}{2}\right)$ functions in the answer! $n=2$, $n=8$

We already know $A_2=3$ and $A_8=0$, therefore, we only need to find B_2 and B_8, all other $B_n=0$

$n=2 \qquad \left[\overset{3}{A_2}\cosh(2\pi)+B_2\sinh(2\pi)\right]\sin(\pi y)=0$

$n=8 \qquad \left[\overset{0}{A_8}\cosh(8\pi)+B_8\sinh(8\pi)\right]\sin(4\pi y)=\sin(4\pi y)$

Thus, $3\cosh(2\pi)+B_2\sinh(2\pi)=0 \Rightarrow \boxed{B_2=-3\frac{\sinh(2\pi)}{\cosh(2\pi)}}$

and $B_8\sinh(8\pi)=1 \Rightarrow \boxed{B_8=\frac{1}{\sinh(8\pi)}}$

Answer: $$u(x,y)=\overbrace{\left[3\cosh(\pi x)-\frac{3\sinh(2\pi)}{\cosh(2\pi)}\sinh(\pi x)\right]\sin(\pi y)}^{n=2}+\overbrace{\left[\frac{1}{\sinh(8\pi)}\sinh(4\pi x)\right]\sin(4\pi y)}^{n=8}$$

Exam #2B, Math 4354
Fall 2018

Name: ANSWERS

Instructions: Turn off all cell phones and electronic devices. No calculators are allowed. Please write neatly and clearly. Put answers in spaces provided. Show all of your work.

1. (12 pts) Compute the eigenvalues λ_n and eigenfunctions $X_n(x)$.

   $X'' + \lambda X = 0$, $X(0) = 0$, $X(\pi) = 0$?

   $\lambda_n = (n)^2$ $\quad X_n(x) = \sin(nx)$, $n = 1, 2, \ldots$

   $X'' + \lambda X = 0$, $X'(0) = 0$, $X'(9) = 0$?

   $\lambda_n = \left(\frac{n\pi}{9}\right)^2$ $\quad X_n(x) = \cos\left(\frac{n\pi x}{9}\right)$, $n = 0, 1, 2, \ldots$

2. (6 pts) Determine whether the PDE is hyperbolic, parabolic, or elliptic.

   (a) $u_t - 6u_{xt} + u = 0$ hyperbolic $\quad B^2 - 4AC = (-6)^2 - 4(0)(0) > 0$

   (a) $u_{xx} + 5u_{yy} + u_y + u = 0$ elliptic $\quad B^2 - 4AC = (0)^2 - 4(1)(5) < 0$

3. (6 pts) Separate variables, $u(x,t) = X(x)T(t)$, for the following PDE: $u_t + 5u_x = u_{xx}$

$$\frac{XT'}{XT} + \frac{5X'T}{XT} = \frac{X''T}{XT} \Rightarrow \frac{T'}{T} + \frac{5X'}{X} = \frac{X''}{X}$$

$$\Rightarrow \boxed{\frac{T'}{T} = \frac{X''}{X} - \frac{5X'}{X}}$$

4. (6 pts) Solve the BVP for the heat equation.
   $u_t = u_{xx}$, $0 < x < 1$, $t > 0$
   $u_x(0,t) = 0$, $u_x(1,t) = 0$, $t > 0$
   $u(x,0) = 8$, $0 < x < 1$.

$$u(x,t) = c_0 + \sum_{n=1}^{\infty} c_n e^{-(n\pi)^2 t} \cos(n\pi x)$$

$$u(x,0) = c_0 + \sum_{n\geq 1}^{\infty} c_n \cos(n\pi x) = 8$$

$c_0 = 8 \quad c_n = 0, \; n = 1, 2, \ldots$

$$\boxed{u(x,t) = 8}$$

5. (20 pts) Solve the BVP for the heat equation.

$u_t = ku_{xx}$, $0 < x < 3$, $t > 0$

$u(0,t) = 0$, $u(3,t) = 0$, $t > 0$. $\quad L=3$

$$u(x,0) = \begin{cases} 2, & 0 < x < 1 \\ 0, & 1 < x < 3. \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi}{3}\right)^2 kt} \sin\left(\frac{n\pi x}{3}\right)$$

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{3}\right) = \begin{cases} 2, & 0<x<1 \\ 0, & 1<x<3 \end{cases}$$

Use Fourier sine coefficients

$$c_n = \frac{2}{3}\int_0^1 2\sin\left(\frac{n\pi x}{3}\right)dx + \frac{2}{3}\int_1^3 0 \cdot \sin\left(\frac{n\pi x}{3}\right)dx$$

$$= -\frac{4 \cdot 3}{3n\pi}\left(\cos\left(\frac{n\pi x}{3}\right)\right)\Big|_0^1 = \frac{-4}{n\pi}\left(\cos\left(\frac{n\pi}{3}\right) - \cos(0)\right)$$

$$= \frac{4}{n\pi}\left(1 - \cos\left(\frac{n\pi}{3}\right)\right)$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{4}{n\pi}\left(1 - \cos\left(\frac{n\pi}{3}\right)\right) e^{-\left(\frac{n\pi}{3}\right)^2 kt} \sin\left(\frac{n\pi x}{3}\right)}$$

6. (20 pts) Solve the BVP for the wave equation.
$u_{tt} = u_{xx},\ 0 < x < 2,\ t > 0$ $L=2$
$u(0,t) = 0,\ u(2,t) = 0,\ t > 0$
$u(x,0) = \sin(2\pi x),\ u_t(x,0) = 4\sin(3\pi x),\ 0 < x < 2.$

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi}{2}t\right) + B_n \sin\left(\frac{n\pi}{2}t\right)\right] \sin\left(\frac{n\pi x}{2}\right)$$

① $u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{2}\right) = \sin(2\pi x)$

② $u_t(x,0) = \sum_{n=1}^{\infty} B_n \cdot \frac{n\pi}{2} \sin\left(\frac{n\pi x}{2}\right) = 4\sin(3\pi x)$

① $\Rightarrow$ $A_4 = 1,\quad n \neq 4,\ A_n = 0$

② $\Rightarrow$ $B_6 \cdot 3\pi = 4,\quad n \neq 6,\ B_6 = 0$
$B_6 = \frac{4}{3\pi}$

$$\boxed{u(x,t) = \cos(2\pi t)\sin(2\pi x) + \frac{4}{3\pi}\sin(3\pi t)\sin(3\pi x)}$$

7. (10 pts) The solution of Laplace's equation on the rectangle $0 < x < 2,\ 0 < y < 3$ is

$$u(x,y) = 2\left[\cosh(\pi x) - \frac{\cosh(2\pi)}{\sinh(2\pi)}\sinh(\pi x)\right]\sin(\pi y) + 7\frac{\sinh(2\pi y)}{\sinh(6\pi)}\sin(2\pi x).$$

What are the four boundary conditions at $x = 0$? $x = 2$? $y = 0$? $y = 3$?

$u(0,y) = 2\sin(\pi y)$
$u(2,y) = 0$
$u(x,0) = 0$
$u(x,3) = 7\sin(2\pi x)$

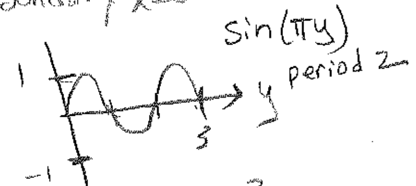
Exam #2B Answers.

8. (20 pts) Solve the BVP for Laplace's equation.
$u_{xx} + u_{yy} = 0,\ 0 < x < 3,\ 0 < y < 3$
$u(0,y) = \sin(\pi y),\ u(3,y) = 2\sin(4\pi y),\ 0 < y < 3$
$u(x,0) = 0,\ u(x,3) = 0,\ 0 < x < 3$ Homogeneous BC in $y \Rightarrow Y'' + \lambda Y = 0$

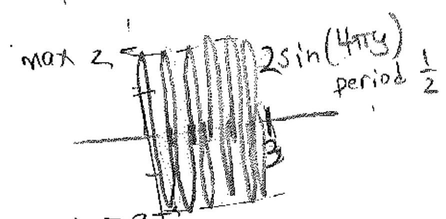
What is the maximum temperature on the rectangle? 2

What is the minimum temperature on the rectangle? −2

Boundary $x=0$



Boundary $x=3$



$X'' - \lambda X = 0$ $\quad$ $\left.\begin{array}{l} Y'' + \lambda Y = 0 \\ Y(0) = 0 \\ Y(3) = 0 \end{array}\right\}$ $\lambda_n = \left(\frac{n\pi}{3}\right)^2$, $\quad Y_n = \sin\left(\frac{n\pi y}{3}\right)$

$X = A_n \cosh\left(\frac{n\pi x}{3}\right) + B_n \sinh\left(\frac{n\pi x}{3}\right)$

$$\boxed{u(x,y) = \sum_{n=1}^{\infty}\left[A_n \cosh\left(\frac{n\pi x}{3}\right) + B_n \sinh\left(\frac{n\pi x}{3}\right)\right]\sin\left(\frac{n\pi y}{3}\right)}$$

Apply the B.C. $x=0,\ x=3$ $\quad \cosh(0)=1, \sinh(0)=0$

$$u(0,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{3}\right) = \sin(\pi y)$$

$\boxed{n=3,\ A_3 = 1 \quad A_n = 0,\ n \neq 3}$

$$u(3,y) = \sum_{n=1}^{\infty}\left[A_n \cosh(n\pi) + B_n \sinh(n\pi)\right]\sin\left(\frac{n\pi y}{3}\right) = 2\sin(4\pi y)$$

Because the BC have $\sin(\pi y)$ and $\sin(4\pi y)$, these are the only $\sin\left(\frac{n\pi y}{3}\right)$ functions in the answer! Also we know all of the A_n coefficients, all we need to find are all of the coefficients for B_n. All $\boxed{B_n = 0}$ for $n \neq 3,\ n \neq 12$

(the coefficients (Why?))

$n=3$ $\quad \left[\overset{1}{A_3}\cosh(3\pi) + B_3 \sinh(3\pi)\right]\sin(\pi y) = \underset{0}{0}$

and $n=12$ $\quad \left[\overset{0}{A_{12}}\cosh(12\pi) + B_{12}\sinh(12\pi)\right]\sin(4\pi y) = \underset{2}{2}\sin(4\pi y)$

$1 \cdot \cosh(3\pi) + B_3 \sinh(3\pi) = 0 \Rightarrow B_3 = -\dfrac{\cosh(3\pi)}{\sinh(3\pi)}$

$0 \cdot \cosh(12\pi) + B_{12}\sinh(12\pi) = 2 \Rightarrow B_{12} = \dfrac{2}{\sinh(12\pi)}$

Answer
$$u(x,y) = \underbrace{\left[\cosh(\pi x) - \frac{\cosh(3\pi)}{\sinh(3\pi)}\sinh(\pi x)\right]\sin(\pi y)}_{n=3} + \underbrace{\frac{2}{\sinh(12\pi)}\cosh(4\pi x)\sin(4\pi y)}_{n=12}$$