Exam # 1A, Math 4354
Fall 2018

Signature: Answers

Instructions: Turn off all cell phones and electronic devices. No calculators are allowed. Please write neatly and clearly. Write answers in spaces provided. Show your work.

1. (25 pts)

15 (a) Consider the set of functions $\{x, x^2, x^3\}$ on $[-1, 1]$. Compute the following:

$$\int_{-1}^{1} x x^2\, dx = \int_{-1}^{1} x^3 dx = \frac{x^4}{4}\Big|_{-1}^{1} = \frac{1}{4} - \frac{(-1)^4}{4} = 0.$$

$$\int_{-1}^{1} x x^3\, dx = \int_{-1}^{1} x^4 dx = \frac{x^5}{5}\Big|_{-1}^{1} = \frac{1}{5} - \frac{(-1)^5}{5} = 2/5.$$

$$\int_{-1}^{1} x^2 x^3\, dx = \int_{-1}^{1} x^5 dx = \frac{x^6}{6}\Big|_{-1}^{1} = \frac{1}{6} - \frac{(-1)^6}{6} = 0$$

Is the set $\{x, x^2, x^3\}$ orthogonal on $[-1, 1]$? Yes or No? No

Exam 1B

5 (b) A general solution of a linear system is $\vec{X}(t) = c_1 e^{-2t}\begin{pmatrix}1\\4\end{pmatrix} + c_2 e^{2t}\begin{pmatrix}1\\-3\end{pmatrix}$.
Is the origin a stable or unstable node, a stable or unstable spiral, a saddle, or a center?

Saddle $\lambda_{1,2} = -2, 2$

Unstable node
$\lambda_{1,2} = 2, 2$

5 (c) A general solution of a linear system is $\vec{X}(t) = c_1\begin{pmatrix}2\cos(3t)\\ \sin(3t)\end{pmatrix} + c_2\begin{pmatrix}-2\sin(3t)\\ \cos(3t)\end{pmatrix}$.
Is the origin a stable or unstable node, a stable or unstable spiral, a saddle, or a center?

Center $\lambda_{1,2} = 0 \pm 3i$

2. (25 pts) Let $\vec{X}'(t) = A\vec{X}(t)$ with $A = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$.

15 (a) Compute the eigenvalues and eigenvectors of matrix A.

10 (b) Write the unique solution of the system if the initial condition is $\vec{X}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

(a) $\lambda^2 - 5\lambda + 6 = 0 \quad (\lambda-2)(\lambda-3) = 0 \quad \boxed{\lambda_{1,2} = 2, 3}$

$\lambda_1 = 2, \; (A - \lambda_1 I)\vec{K}_1 = \begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{K}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = 3, \; (A - \lambda_2 I)\vec{K}_2 = \begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{K}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

(b) $\vec{X}(t) = c_1 e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\vec{X}(0) = \begin{pmatrix} c_1 + c_2 \\ c_1 + 2c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \Rightarrow \left.\begin{matrix} c_1 + c_2 = 1 \\ c_1 + 2c_2 = 3 \end{matrix}\right\} \Rightarrow \underline{c_2 = 2}, \underline{c_1 = -1}$

$$\boxed{\begin{aligned} &\vec{X}(t) = -e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ \text{or} \quad &x(t) = -e^{2t} + 2e^{3t} \\ &y(t) = -e^{2t} + 4e^{3t} \end{aligned}}$$

Exam 1B $\vec{X}(t) = -e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 3e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\quad \left.\begin{matrix} c_1 + c_2 = 2 \\ c_1 + 2c_2 = 5 \end{matrix}\right\} \Rightarrow c_2 = 3, c_1 = -1$

or

$x(t) = -e^{2t} + 3e^{3t}$

$y(t) = -e^{2t} + 6e^{3t}$

3. (25 pts) For the nonlinear autonomous system $dx/dt = 8 - 2x - y$ and $dy/dt = y(2 - x)$,

8 (a) Compute the two critical points.

7 (b) Compute the Jacobian matrix.

10 (c) Use the Jacobian matrix evaluated at each critical point to classify each critical point as a stable or unstable node, a stable or unstable spiral, or a saddle.

$$\left.\begin{array}{l} y(2-x)=0 \Rightarrow y=0 \text{ or } x=2 \\ 8-2x-y=0 \Rightarrow y=8-2x \end{array}\right\} \Rightarrow \begin{array}{c} y=0 \text{ and } x=4 \\ \text{or} \\ x=2 \text{ and } y=4 \end{array}$$

(a) Critical points $(2,4)$ and $(4,0)$

(b) $J(x,y) = \begin{pmatrix} -2 & -1 \\ -y & 2-x \end{pmatrix}$

(c) at $(4,0)$ $J(4,0) = \begin{pmatrix} -2 & -1 \\ 0 & -2 \end{pmatrix}$ $\lambda_{1,2} = -2, -2$

$(4,0)$ is a stable node

at $(2,4)$ $J(2,4) = \begin{pmatrix} -2 & -1 \\ -4 & 0 \end{pmatrix}$

$\lambda^2 + 2\lambda - 4 = 0 \Rightarrow \lambda = \frac{-2 \pm \sqrt{4+16}}{2} = \frac{-2}{2} \pm \frac{2\sqrt{5}}{2}$

$\lambda_{1,2} = -1 \pm \sqrt{5}$

$(2,4)$ is a saddle

Exam 1B (a) Critical points $(5,0), (2,6)$

(c) At $(5,0)$ $J(5,0) = \begin{pmatrix} -2 & -1 \\ 0 & -3 \end{pmatrix}$ $\lambda_{1,2} = -2, -3$ stable node

At $(2,6)$ $J(2,6) = \begin{pmatrix} -2 & -1 \\ -6 & 0 \end{pmatrix}$ $\lambda_{1,2} = -1 \pm \sqrt{7}$ saddle

$\lambda^2 + 2\lambda - 6 = 0$

4. (25 pts) For the function $f(x) = \begin{cases} 0, & -\pi < x < 0, \\ 3, & 0 < x < \pi, \end{cases}$

15 (a) Compute the Fourier coefficients, a_0, a_n, and b_n, then write the Fourier series for $f(x)$.

10 (b) Graph the Fourier series on the interval $[-3\pi, 3\pi]$. To what value does the Fourier series converge

at $x = 0$? $\underline{3/2}$ at $x = \pi/2$? $\underline{3}$, at $x = 2\pi$? $\underline{3/2}$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} \cancel{0\,dx}^{\,0} + \frac{1}{\pi}\int_0^{\pi} 3\,dx = 3$$

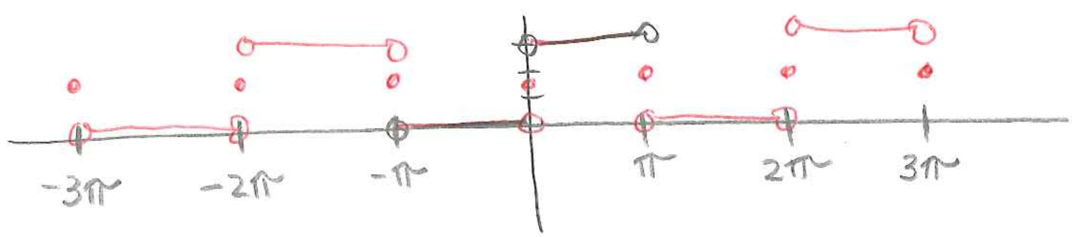
$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} \cancel{0\cos(nx)\,dx}^{\,0} + \frac{1}{\pi}\int_0^{\pi} 3\cos(nx)\,dx = \frac{3}{n\pi}\sin(nx)\Big|_0^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} 0.\sin(nx)\,dx + \frac{1}{\pi}\int_0^{\pi} 3\sin(nx)\,dx = \frac{3}{n\pi}(-\cos(nx))\Big|_0^{\pi}$$

$$= \frac{3}{n\pi}\left(-\cos(n\pi) + \cos(0)\right) = \frac{3}{n\pi}\left(1 - \cos(n\pi)\right) = \frac{3}{n\pi}\left(1 - (-1)^n\right).$$

(a) $$\boxed{f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{3}{n\pi}\left(1 - (-1)^n\right)\sin(nx)}$$

(b)



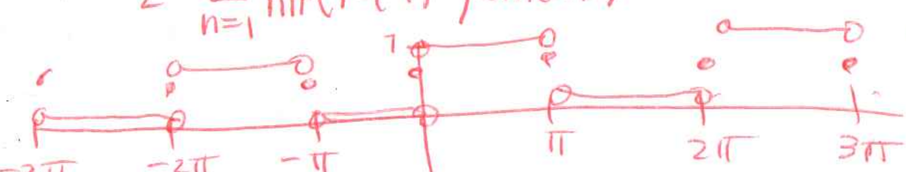
At $x = 0$? $\underline{3/2}$ At $x = \pi/2$? $\underline{3}$ At $x = 2\pi$? $\underline{3/2}$

Exam 1B

(a) $a_0 = 7$, $a_n = 0$, $b_n = \frac{7}{n\pi}(1-(-1)^n)$

$$f(x) = \frac{7}{2} + \sum_{n=1}^{\infty} \frac{7}{n\pi}\left(1-(-1)^n\right)\sin(nx)$$

(b)



At $x=0$? $7/2$

$x = \pi/2$? 7

$x = 2\pi$? $7/2$