Exam #3B Math 4354
Fall 2018

Name: Answers

Instructions: Turn off all cell phones and electronic devices. No calculators are allowed. Please write neatly and clearly. Put answers in spaces provided or circle your answers. Show all of your work for full credit.

1. (20 pts) The steady-state temperature on a circular plate, $u(r, \theta)$, is the solution of $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$. Find the steady-state temperature if the plate has radius $r = 4$ and the temperature on the circumference is $u(4, \theta) = 2 - 6\cos(3\theta)$. General Solution: $A_0 + \sum_{n=1}^{\infty} \left(\frac{r}{c}\right)^n \left[A_n \cos(n\theta) + B_n \sin(n\theta)\right]$

   Therefore, we can write the solution from the temperature on the boundary.

   $u(r, \theta) =$ $2 - 6\left(\frac{r}{4}\right)^n \cos(n\theta)$

   What is the maximum temperature on the plate? $2+6=8$ (when $\cos(\theta)=-1$)

   What is the minimum temperature on the plate? $2-6=-4$ (when $\cos(\theta)=1$)

2. (15 pts) Solve the BVP for the heat equation on a rectangle.

   PDE: $u_t = k(u_{xx} + u_{yy})$, $0 < x < 1$, $0 < y < 1$, $t > 0$ $a=1, b=1$
   BC: $u(0, y, t) = 0$, $u(1, y, t) = 0$, $0 < y < 1$, $t > 0$
   BC: $u(x, 0, t) = 0$, $u(x, 1, t) = 0$, $0 < x < 1$, $t > 0$
   IC: $u(x, y, 0) = 4\sin(2\pi x)\sin(\pi y) + \sin(\pi x)\sin(5\pi y)$, $0 < x < 1$, $0 < y < 1$.

   $\lambda_m = \left(\frac{m\pi}{a}\right)^2 = (m\pi)^2$

   $\mu_n = \left(\frac{n\pi}{b}\right)^2 = (n\pi)^2$

   General solution is

   $$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} e^{-kt\left[(m\pi)^2 + (n\pi)^2\right]} \sin(m\pi x)\sin(n\pi y)$$

   When $t=0$

   $$u(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin(m\pi x)\sin(n\pi y) = 4\sin(2\pi x)\sin(\pi y) + \sin(\pi x)\sin(5\pi y)$$

   only two $A_{mn} \neq 0$, $A_{21} = 4$, $A_{15} = 1$

   $$u(x,y,t) = 4e^{-kt\left[(2\pi)^2 + (\pi)^2\right]} \sin(2\pi x)\sin(\pi y) + e^{-kt\left[(\pi)^2 + (5\pi)^2\right]} \sin(\pi x)\sin(5\pi y)$$

   $$\boxed{u(x,y,t) = 4e^{-5\pi^2 kt}\sin(2\pi x)\sin(\pi y) + e^{-26\pi^2 kt}\sin(\pi x)\sin(5\pi y)}$$

3. (15 pts) Solve the BVP for the heat equation with Fourier transforms.

PDE: $u_t = u_{xx}$, $0 < x < \infty$, $t > 0$
BC: $u_x(0,t) = 0$, $t > 0$
IC: $u(x,0) = xe^{-2x}$, $0 < x < \infty$.

Since $u_x(0,t)=0$, apply Fourier cosine transforms.

$\mathcal{F}_c\{u_t\} = \mathcal{F}_c\{u_{xx}\} \Rightarrow \frac{dU}{dt} = -\alpha^2 U - \overset{0}{\cancel{u_x(0,t)}} \Rightarrow \frac{dU}{dt} = -\alpha^2 U \Rightarrow \boxed{U(\alpha,t) = ce^{-\alpha^2 t}}$

Transform the initial data to find c.

$\mathcal{F}_c\{u(x,0)\} = \mathcal{F}_c\{xe^{-2x}\} \Rightarrow U(\alpha,0) = \frac{4-\alpha^2}{(\alpha^2+4)^2} \Rightarrow U(\alpha,0) = c = \frac{4-\alpha^2}{(\alpha^2+4)^2}$

$\boxed{U(\alpha,t) = \frac{4-\alpha^2}{(\alpha^2+4)^2} e^{-\alpha^2 t}}$ Now take the inverse Fourier cosine transform.

$\mathcal{F}_c^{-1}\{U(\alpha,t)\} = \mathcal{F}_c^{-1}\left\{\frac{4-\alpha^2}{(\alpha^2+4)^2} e^{-\alpha^2 t}\right\} \Rightarrow \boxed{u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{4-\alpha^2}{(\alpha^2+4)^2} e^{-\alpha^2 t} \cos(\alpha x)\, d\alpha}$

4. (20 pts) Solve the BVP for the wave equation with Fourier transforms.

PDE: $u_{tt} = u_{xx}$, $0 < x < \infty$, $t > 0$
BC: $u(0,t) = 0$, $t > 0$
IC: $u(x,0) = 0$ and $\left.\frac{\partial u}{\partial t}\right|_{t=0} = 8e^{-x}$, $0 < x < \infty$.

Since $u(0,t)=0$, apply Fourier sine transforms.

$\mathcal{F}_s\{u_{tt}\} = \mathcal{F}_s\{u_{xx}\} \Rightarrow \frac{d^2U}{dt^2} = -\alpha^2 U + \alpha \overset{0}{\cancel{u(0,t)}} \Rightarrow \frac{d^2U}{dt^2} + \alpha^2 U = 0$

$\Rightarrow \boxed{U(\alpha,t) = c_1\cos(\alpha t) + c_2\sin(\alpha t)}$

Transform the initial data to find c_1 and c_2.

$\mathcal{F}_s\{u(x,0)\} = \mathcal{F}_s\{0\} \Rightarrow U(\alpha,0) = 0$

$\mathcal{F}_s\left\{\left.\frac{\partial u}{\partial t}\right|_{t=0}\right\} = \mathcal{F}_s\{8e^{-x}\} \Rightarrow \left.\frac{dU}{dt}\right|_{t=0} = \frac{8\alpha}{\alpha^2+1}$

$U(\alpha,0) = c_1\cos(0) + c_2\sin(0) = c_1 = 0$

$\left.\frac{dU}{dt}\right|_{t=0} = \left. c_1\alpha\sin(\alpha t) + c_2\alpha\cos(\alpha t)\right|_{t=0} = c_1\alpha\sin(0) + c_2\alpha\cos(0) = c_2\alpha = \frac{8\alpha}{\alpha^2+1}$

$\Rightarrow c_2 = \frac{8}{\alpha^2+1} \Rightarrow \boxed{U(\alpha,t) = \frac{8}{\alpha^2+1}\sin(\alpha t)}$ Take inverse Fourier sine transform

$\mathcal{F}_s^{-1}\{U(\alpha,t)\} = \mathcal{F}_s^{-1}\left\{\frac{8}{\alpha^2+1}\sin(\alpha t)\right\} \Rightarrow \boxed{u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{8}{\alpha^2+1}\sin(\alpha t)\sin(\alpha x)\, d\alpha}$

5. (25 pts) Solve the nonhomogeneous BVP for the heat equation.

PDE: $u_t = u_{xx} + 3\sin(3x)$, $0 < x < \pi$, $t > 0$.
BC: $u(0,t) = 0$, $u(\pi,t) = 0$, $t > 0$
IC: $u(x,0) = 0$, $0 < x < \pi$.

Steady-state temperature: $\psi(x) = \frac{1}{3}\sin(3x)$

$u(x,t) = -\frac{1}{3}e^{-9t}\sin(3x) + \frac{1}{3}\sin(3x)$

$u(x,t) = \underbrace{v(x,t)}_{\text{homogeneous}} + \underbrace{\psi(x)}_{\text{Steady-State}}$

$\begin{cases} v_t = v_{xx} & 0<x<\pi \\ v(0,t)=0, \; v(\pi,t)=0 \end{cases}$ $\lambda_n = \left(\frac{n\pi}{L}\right)^2 = (n)^2$ $X_n = \sin(nx)$

$\begin{cases} \psi'' + 3\sin(3x) = 0 & \psi(0)=0, \; \psi(\pi)=0 \end{cases}$

$\psi'' = -3\sin(3x)$

$\psi' = \cos(3x) + C_1$

$\psi = \frac{1}{3}\sin(3x) + C_1x + C_2$

$\psi(0) = C_2 = 0$

$\psi(\pi) = \frac{1}{3}\cancel{\sin(3\pi)} + C_1\pi = 0 \Rightarrow C_1 = 0$

$\psi(x) = \frac{1}{3}\sin(3x)$

General Solution:

$v(x,t) = \sum_{n=1}^{\infty} A_n e^{-(n^2t)}\sin(nx)$

$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(n^2t)}\sin(nx) + \psi(x)$

$u(x,0) = 0 = \sum_{n=1}^{\infty} A_n\sin(nx) + \psi(x) \Rightarrow \sum_{n=1}^{\infty} A_n\sin(nx) = -\frac{1}{3}\sin(3x)$

$\Rightarrow A_3 = -\frac{1}{3}$ $A_n = 0, n \neq 3.$

$$\boxed{u(x,t) = -\frac{1}{3}e^{-9t}\sin(3x) + \frac{1}{3}\sin(3x)}$$

6. (5 pts) How does the solution in Problem 5 change if there is lateral heat loss?

PDE: $u_t = u_{xx} - hu + 3\sin(3x)$, $0 < x < \pi$, $t > 0$.

Homogeneous Solution is a solution

$\begin{cases} v_t = v_{xx} - hu \\ v(0,t)=0, \; v(\pi,t)=0 \end{cases}$

General Solution:

$\boxed{v(x,t) = \sum_{n=1}^{\infty} A_n e^{-ht} e^{-n^2t}\sin_3(nx)}$

Steady-State is a solution

$\begin{cases} \psi'' - h\psi + 3\sin(3x) = 0 \\ \psi(0)=0 \quad \psi(\pi)=0 \end{cases}$

$\psi(x) = \frac{3}{9+h}\sin(3x)$

$u(x,t) = -\frac{3}{9+h}e^{-(9+h)t}\sin(3x) + \frac{3}{9+h}\sin(3x)$

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   $u(r,\theta) =$ $2 - 8\left(\frac{r}{3}\right)^n \cos(4\theta)$

   What is the maximum temperature on the plate? $2-(8)(-1)=10$

   What is the minimum temperature on the plate? $2-8(1) = -6$

2. (15 pts) Solve the BVP for the heat equation on a rectangle.

   PDE: $u_t = k(u_{xx} + u_{yy})$, $0 < x < 1$, $0 < y < 1$, $t > 0$

   BC: $u(0,y,t) = 0$, $u(1,y,t) = 0$, $0 < y < 1$, $t > 0$ $a=1, b=1,$

   BC: $u(x,0,t) = 0$, $u(x,1,t) = 0$, $0 < x < 1$, $t > 0$

   IC: $u(x,y,0) = 2\sin(3\pi x)\sin(\pi y) - \sin(\pi x)\sin(2\pi y)$, $0 < x < 1$, $0 < y < 1$.

   $\lambda_m = \left(\frac{m\pi}{a}\right)^2 = (m\pi)^2$

   $\mu_n = \left(\frac{n\pi}{b}\right)^2 = (n\pi)^2$

   General Solution: $u(x,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} e^{-k[(m\pi)^2+(n\pi)^2]t} \sin(m\pi x)\sin(n\pi y)$

   $$\boxed{u(x,y,t) = 2e^{-k[(3\pi)^2+(\pi)^2]t}\sin(3\pi x)\sin(\pi y) - e^{-k[(\pi)^2+(2\pi)^2]t}\sin(\pi x)\sin(2\pi y)}$$

   $$\boxed{u(x,y,t) = 2e^{-k(10\pi^2)t}\sin(3\pi x)\sin(\pi y) - e^{-k(5\pi^2)t}\sin(\pi x)\sin(2\pi y)}$$

3. (15 pts) Solve the BVP for the heat equation with Fourier transforms.

PDE: $u_t = u_{xx}$, $0 < x < \infty$, $t > 0$
BC: $u_x(0,t) = 0$, $t > 0$
IC: $u(x,0) = xe^{-3x}$, $0 < x < \infty$.

Apply Fourier Cosine Transforms

$$\mathcal{F}_c\{u_t\} = \mathcal{F}_c\{u_{xx}\} \Rightarrow \frac{dU}{dt} = -\alpha^2 U - \cancel{u_x(0,t)}^{\to 0} \Rightarrow \boxed{U(\alpha,t) = ce^{-\alpha^2 t}}$$

Transform initial data to find c

$$\mathcal{F}_c\{u(x,0)\} = \mathcal{F}_c\{xe^{-3x}\} \Rightarrow U(\alpha,0) = \frac{9-\alpha^2}{(\alpha^2+9)^2} = c$$

$$\boxed{U(\alpha,t) = \frac{9-\alpha^2}{(\alpha^2+9)^2} e^{-\alpha^2 t}}$$

Take inverse Fourier cosine transform.

$$\mathcal{F}_c^{-1}\{U(\alpha,t)\} = \mathcal{F}_c^{-1}\left\{\frac{9-\alpha^2}{(\alpha^2+9)^2} e^{-\alpha^2 t}\right\}$$

$$\boxed{u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{9-\alpha^2}{(\alpha^2+9)^2} e^{-\alpha^2 t}\cos(\alpha x)\, d\alpha}$$

4. (20 pts) Solve the BVP for the wave equation with Fourier transforms.

PDE: $u_{tt} = u_{xx}$, $0 < x < \infty$, $t > 0$
BC: $u(0,t) = 0$, $t > 0$
IC: $u(x,0) = 0$ and $\left.\frac{\partial u}{\partial t}\right|_{t=0} = 2e^{-x}$, $0 < x < \infty$.

Apply Fourier sine transforms

$$\mathcal{F}_s\{u_{tt}\} = \mathcal{F}_s\{u_{xx}\} \Rightarrow \frac{d^2U}{dt^2} = -\alpha^2 U + \alpha \cancel{u(0,t)}^{\to 0} \Rightarrow \frac{d^2U}{dt^2} + \alpha^2 U = 0$$

$$\boxed{U(\alpha,t) = c_1\cos(\alpha t) + c_2\sin(\alpha t)}$$

Transform initial conditions to find c_1 and c_2

$$\mathcal{F}_s\{u(x,0)\} = \mathcal{F}_s\{0\} \Rightarrow U(\alpha,0) = 0 = c_1\cos(0) + c_2\sin(0) = c_1 \Rightarrow c_1 = 0$$

$$\mathcal{F}_s\left\{\left.\frac{\partial u}{\partial t}\right|_{t=0}\right\} = \mathcal{F}_s\{2e^{-x}\} \Rightarrow \left.\frac{dU}{dt}\right|_{t=0} = \frac{2\alpha}{\alpha^2+1} = -c_1\alpha\sin(0) + c_2\alpha\cos(0) = c_2\alpha$$

$$\Rightarrow c_2 = \frac{2}{\alpha^2+1}$$

$$\boxed{U(\alpha,t) = \frac{2}{\alpha^2+1}\sin(\alpha t)}$$

Take inverse Fourier sine transforms

$$\mathcal{F}_s^{-1}\{U(\alpha,t)\} = \mathcal{F}_s^{-1}\left\{\frac{2}{\alpha^2+1}\sin(\alpha t)\right\} \Rightarrow \boxed{u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{2}{\alpha^2+1}\sin(\alpha t)\sin(\alpha x)\, d\alpha}$$

5. (25 pts) Solve the nonhomogeneous BVP for the heat equation.

PDE: $u_t = u_{xx} + 2\sin(2x)$, $0 < x < \pi$, $t > 0$.
BC: $u(0,t) = 0$, $u(\pi,t) = 0$, $t > 0$
IC: $u(x,0) = 0$, $0 < x < \pi$.

Steady-state temperature: $\psi(x) =$ $\frac{1}{2}\sin(2x)$

$u(x,t) =$ $-\frac{1}{2}e^{-4t}\sin(2x) + \frac{1}{2}\sin(2x)$

$$u(x,t) = \underbrace{v(x,t)}_{\text{homogeneous}} + \underbrace{\psi(x)}_{\text{steadystate}}$$

$$\begin{cases} v_t = v_{xx} & 0 < x < \pi \\ v(0,t)=0, v(\pi,t)=0 \end{cases}$$

General Solution:

$$v(x,t) = \sum_{n=1}^{\infty} A_n e^{-(n^2)t} \sin(nx)$$

$$\begin{cases} \psi'' + 2\sin(2x) = 0 \\ \psi(0)=0 \quad \psi(\pi)=0 \end{cases}$$

$\psi''(x) = -2\sin(2x)$

$\psi' = \cos(2x) + c_1$

$\psi = \frac{1}{2}\sin(2x) + c_1 x + c_2$

$\psi(0) = c_2 = 0$

$\psi(\pi) = c_1\pi = 0, c_1 = 0$

$\psi(x) = \frac{1}{2}\sin(2x)$

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-n^2 t}\sin(nx) + \frac{1}{2}\sin(2x)$$

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin(nx) + \frac{1}{2}\sin(2x) = 0 \Rightarrow \sum_{n=1}^{\infty} A_n \sin(nx) = -\frac{1}{2}\sin(2x)$$

$A_2 = -\frac{1}{2}$, $A_n = 0$, $n \neq 2$

$$\boxed{u(x,t) = -\frac{1}{2}e^{-4t}\sin(2x) + \frac{1}{2}\sin(2x)}$$

6. (5 pts) How does the solution in Problem 5 change if there is lateral heat loss?

PDE: $u_t = u_{xx} - hu + 2\sin(2x)$, $0 < x < \pi$, $t > 0$.

Homogeneous

$$\begin{cases} v_t = v_{xx} - hu \\ v(0,t)=0 \quad v(\pi,t)=0 \end{cases}$$

$$\boxed{v(x,t) = \sum_{n=1}^{\infty} A_n e^{-n^2 t} e^{-ht} \sin(nx)}$$

Steady-State

$$\begin{cases} \psi'' - hu + 2\sin(2x) = 0 \\ \psi(0)=0 \\ \psi(\pi)=0 \end{cases}$$

$\psi(x) = \frac{2}{4+h}\sin(2x)$

$$u(x,t) = -\frac{2}{4+h}e^{-(4+h)t}\sin(2x) + \frac{2}{4+h}\sin(2x)$$