Review Exam # 2, Math 4354
Chapter 11.4, 12.1-12.5

Topics: **Eigenvalue Problems:** Find **eigenvalues** λ (negative, positive, or zero) and nonzero **eigenfunctions** $X_n(x)$ of a second-order linear ODE with two HOMOGENEOUS boundary conditions.

I. $X'' + \lambda X = 0$, $0 < x < L$, $X(0) = 0$, $X(L) = 0$, $X_n(x) = \sin(\sqrt{\lambda_n}x)$, $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $n = 1, 2, 3, \ldots$

II. $X'' + \lambda X = 0$, $0 < x < L$, $X(0) = 0$, $X'(L) = 0$, $X_n(x) = \sin(\sqrt{\lambda_n}x)$, $\lambda_n = \left(\frac{2n-1}{2L}\pi\right)^2$, $n = 1, 2, 3, \ldots$ Exam3

III. $X'' + \lambda X = 0$, $0 < x < L$, $X'(0) = 0$, $X(L) = 0$, $X_n(x) = \cos(\sqrt{\lambda_n}x)$, $\lambda_n = \left(\frac{2n-1}{2L}\pi\right)^2$, $n = 1, 2, 3, \ldots$

IV. $X'' + \lambda X = 0$, $0 < x < L$, $X'(0) = 0$, $X'(L) = 0$, $X_n(x) = \cos(\sqrt{\lambda_n}x)$, $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $n = 0, 1, 2, 3, \ldots$

The set of eigenfunctions $\{X_n(x)\}$ is orthogonal on $[0, L]$, $\int_0^L X_n(x)X_m(x)\,dx = 0$ for $m \neq n$.

Boundary value problems: Linear second-order PDE in $u(x, y)$: $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$ where A, B, C, D, E, F, and G are functions of x and y. The linear PDE is **homogeneous** if $G = 0$. To classify the 2nd order PDE only the coefficients of the second-order derivatives are important: A, B, C:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + \cdots = 0$$

hyperbolic if $B^2 - 4AC > 0$, **parabolic** if $B^2 - 4AC = 0$ or **elliptic** if $B^2 - 4AC < 0$.

Separation of Variables for $u(x, y)$, assume $u(x, y) = X(x)Y(y)$ and separate variable to solve two ODE problems. The solutions are called separable solutions or product solutions. Superposition of all solutions of a homogeneous linear PDE (linear combination of solutions) yields a general solution, $u(x, y) = \sum c_n u_n(x, y) = \sum X_n(x)Y_n(x)$.

Heat Equation: $u_t = ku_{xx}$, $0 < x < L$, $t > 0$. Solution $u(x, t)$ is the temperature of a rod of length L where the lateral surface is insulated (no heat flux from the sides), $k > 0$ is thermal diffusivity, a parabolic PDE. Applying separation of variables, $u(x, t) = X(x)T(t)$: $T' + \lambda kT = 0$ and $X'' + \lambda X = 0$.

$$\boxed{u(x, t) = c_0 + \sum_{n=1}^{\infty} c_n e^{-\lambda_n kt} X_n(x)}$$

For example, Dirichlet BC: $u(0, t) = 0 = u(L, t)$, $u(x, t) = \sum_{n=1}^{\infty} c_n e^{-(n\pi/L)^2 kt} \sin\left(\frac{n\pi x}{L}\right)$

The coefficients c_n depend on initial temperature, $u(x, 0) = f(x)$, and sometimes can be found by "matching coefficients". Otherwise

$$c_0 = \frac{1}{L}\int_0^L f(x)\,dx, \quad c_n = \frac{2}{L}\int_0^L f(x)X_n(x)\,dx, \; n = 1, 2, \ldots$$

Wave equation: $u_{tt} = a^2u_{xx}$, $0 < x < L$, $t > 0$. Solution $u(x, t)$ is the displacement from zero of a string at position x and time t, hyperbolic PDE. Separation of variables $u(x, t) = X(x)T(t)$ yields two ODEs: $T'' + a^2\lambda T = 0$, $X'' + \lambda X = 0$, $\lambda = \alpha^2$.

$$\boxed{u(x, t) = a_0 + b_0 t + \sum_{n=1}^{\infty}[a_n \cos(\sqrt{\lambda_n}t) + b_n \sin(a\sqrt{\lambda_n}t)]X_n(x)}.$$

The coefficients a_n and b_n are calculated from initial position and initial velocity, $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$, and sometimes can be found by "matching coefficients" or

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx, \quad a_n = \frac{2}{L}\int_0^L f(x)X_n(x)\,dx, \; n = 1, 2, \ldots$$

$$b_0 = \frac{1}{L}\int_0^L g(x)\,dx, \quad b_n a\sqrt{\lambda_n} = \frac{2}{L}\int_0^L g(x)X_n(x)\,dx, \; n = 1, 2, \ldots$$

Laplace's equation: $u_{xx} + u_{yy} = 0$, $0 < x < b$, $0 < y < c$. Solution $u(x, y)$ is the steady-state temperature on a rectangular plate, elliptic PDE. The solution from separation of variables $u(x, y) = X(x)Y(y)$ gives either $X'' + \lambda X = 0$, $Y'' - \lambda Y = 0$ or $Y'' + \lambda Y = 0$, $X'' - \lambda X = 0$, to yield one of the forms:

$$\boxed{u(x, y) = a_0 + b_0 y + \sum_{n=1}^{\infty}[a_n \cosh(\sqrt{\lambda_n}y) + b_n \sinh(\sqrt{\lambda_n}y)]X_n(x) \;\text{ or }\; u(x, y) = a_0 + b_0 x + \sum_{n=1}^{\infty}[a_n \cosh(\sqrt{\lambda_n}x) + b_n \sinh(\sqrt{\lambda_n}x)]Y_n(y)}$$

where either $X_n(x)$ or $Y_n(y)$ are calculated from HOMOGENEOUS BCs (see eigenfunctions above). The coefficients a_n and b_n are calculated from the NONhomogeneous BCs and sometimes can be found by "matching coefficients". Otherwise, coefficients can be found using the orthogonality of the eigenfunctions.

NOTE: $c_0 = 0$, $a_0 = 0$, $b_0 = 0$ in cases I, II, III. Only in case IV, they may be nonzero because $\lambda_0 = 0$, $X_0(x) = 1$.

Review for EXAM # 2: Webwork 3, 4, Written Assignments, Handouts

Practice Problems:

1. Solve the heat, wave, or Laplace's equation. Express the coefficients in terms of $f(x)$ and $g(x)$ in (d).
   (a) $u_t = 4u_{xx}$, $0 < x < \pi$, $u_x(0,t) = 0 = u_x(\pi,t)$, $t > 0$ and $u(x,0) = 3 + 5\cos(2x)$, $0 < x < \pi$.
   (b) $u_{tt} = u_{xx}$, $0 < x < 1$, $u(0,t) = 0 = u(1,t)$, $t > 0$, $u(x,0) = 4\sin(\pi x)$ and $u_t(x,0) = 6\sin(2\pi x)$, $0 < x < 1$.
   (c) $u_{xx} + u_{yy} = 0$, $0 < x < 1$, $0 < y < 1$, $u(0,y) = 0 = u(1,y)$, $0 < y < 1$, $u(x,0) = 7\sin(\pi x) - \sin(3\pi x)$ and $u(x,1) = 0$, $0 < x < 1$.
   Exam 3 { (d) $u_{tt} = u_{xx}$, $0 < x < 1$, $\underline{u(0,t) = 0 = u_x(1,t)}$, $t > 0$, $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$, $0 < x < 1$.
2. Solve the heat equation $u_t = u_{xx}$, $0 < x < \pi$ for $u(0,t) = 0 = u(\pi,t)$, $t > 0$, and $u(x,0) = 2$, $0 < x < \pi$.
3. The solution of the wave equation $u_{tt} = u_{xx}$ on $0 < x < \pi$, $t > 0$ is $u(x,t) = 4\sin(3t/2)\cos(3x/2)$. Find the $\underline{\text{homogeneous}}$ boundary conditions at $x = 0$ and $x = \pi$ and the initial position and initial velocity $u(x,0)$, $u_t(x,0)$.

1 (a) $k=4$, $L=\pi$ $\quad T'+4\lambda T=0$, $T=e^{-4\lambda t}$ $\quad X''+\lambda X=0$, $X'(0)=0=X'(\pi)$ $\quad \lambda_n=\left(\frac{n\pi}{\pi}\right)^2=n^2$, $n=0,1,2,\dots$ $\quad X_n(x)=\cos(nx)$, $n=0,1,2,\dots$

$* \; u(x,t) = c_0 + \sum_{n=1}^{\infty} c_n e^{-4(n^2)t}\cos(nx) = c_0 + c_1 e^{-4t}\cos(x) + c_2 e^{-4(4)t}\cos(2x) + c_3 e^{-4(9)t}\cos(3x) + \dots$

$u(x,0) = c_0 + c_1\cos(x) + c_2\cos(2x) + c_3\cos(3x) + \dots$

$= 3 + 5\cos(2x) \Rightarrow c_0 = 3,\ c_1 = 0,\ c_2 = 5,\ c_n = 0,\ n \geq 3.$

$\boxed{u(x,t) = 3 + 5e^{-16t}\cos(2x)}$

(b) $a^2=1$, $L=1$ $\quad T''+\lambda T=0$, $X''+\lambda X=0$ $\quad X(0)=0=X(1)$, $\lambda_n=\left(\frac{n\pi}{1}\right)^2=(n\pi)^2$, $n=1,2,\dots$, $X_n(x)=\sin(n\pi x)$, $n=1,2,\dots$

$T = c_1\cos(\sqrt{\lambda}t) + c_2\sin(\sqrt{\lambda}t)$

$* \; u(x,t) = \sum_{n=1}^{\infty}\left[a_n\cos(n\pi t) + b_n\sin(n\pi t)\right]\sin(n\pi x)$

$u(x,0) = \sum a_n \sin(n\pi x) = 4\sin(\pi x) \Rightarrow a_1 = 4,\ a_n = 0,\ n = 1,2,\dots$

$u_t(x,t) = \sum_1^{\infty}\left[-a_n n\pi\sin(n\pi t) + b_n n\pi\cos(n\pi t)\right]\sin(n\pi x)$

$u_t(x,0) = \sum + b_n n\pi\sin(n\pi x) = 6\sin(2\pi x) \qquad b_2\cdot 2\pi = 6 \Rightarrow b_2 = \frac{3}{\pi},\ b_n = 0,\ n \neq 2$

$\boxed{u(x,t) = 4\cos(\pi t)\sin(\pi x) + \frac{3}{\pi}\sin(2\pi t)\sin(2\pi x)}$

(c) 

$Y''-\lambda Y=0 \quad X''+\lambda X=0,\ X(0)=0\ X(1)=1 \quad \lambda_n=\left(\frac{n\pi}{1}\right)^2,\ n=1,2,\dots\ X_n(x)=\sin(n\pi x)$

$Y = c_1\cosh(\sqrt{\lambda_n}y) + c_2\sinh(\sqrt{\lambda_n}y)$

$* \; u(x,y) = \sum_{n=1}^{\infty}\left[a_n\cosh(n\pi y) + b_n\sinh(n\pi y)\right]\sin(n\pi x)$

B.C only y: $u(x,0) = \sum_{n=1}^{\infty} a_n\sin(n\pi x) = 7\sin(\pi x) - \sin(3\pi x) \Rightarrow a_1 = 7,\ a_3 = -1,\ a_n = 0$, OTW.

$u(x,1) = \sum_{n=1}^{\infty}\left[a_n\cosh(n\pi) + b_n\sinh(n\pi)\right]\sin(n\pi x) = 0 \qquad b_n = 0,\ n \neq 1,3$

$= \left[7\cosh(\pi) + b_1\sinh(\pi)\right]\sin(\pi x) + \left[-\cosh(3\pi) + b_3\sinh(3\pi)\right]\sin(3\pi x)$

$\Rightarrow b_1 = -\frac{7\cosh(\pi)}{\sinh(\pi)},\quad b_3 = \frac{\cosh(3\pi)}{\sinh(3\pi)}$

$$\boxed{u(x,y) = \left[7\cosh(\pi y) - \frac{7\cosh(\pi)}{\sinh(\pi)}\sinh(\pi y)\right]\sin(\pi x) + \left[-\cosh(3\pi y) + \frac{\cosh(3\pi)}{\sinh(3\pi)}\sinh(3\pi y)\right]\sin(3\pi x)}$$

Exam 3 { (d) $a^2=1$, $L=1$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty}\left[a_n\cos\left(\tfrac{2n-1}{2}\pi t\right) + b_n\sin\left(\tfrac{2n-1}{2}\pi t\right)\right]\sin\left(\tfrac{2n-1}{2}\pi x\right)}$$

$$a_n = \frac{2}{1}\int_0^1 f(x)\sin\left(\tfrac{2n-1}{2}\pi x\right)dx \qquad b_n\cdot\frac{2n-1}{2}\pi = \frac{2}{1}\int_0^1 g(x)\sin\left(\tfrac{2n-1}{2}\pi x\right)dx$$

3. $u(x,t) = 4\sin\left(\frac{3t}{2}\right)\cos\left(\frac{3x}{2}\right)$

$u_x(x,t) = -4\sin\left(\frac{3t}{2}\right)\frac{3}{2}\sin\left(\frac{3x}{2}\right)$

$\underline{\text{Homogeneous BC}}$: $\boxed{u_x(0,t)=0,\ u(\pi,t)=0}$ $\quad$ IC: $u(x,0)=0$, $u_t(x,0) = 6\cos\left(\frac{3x}{2}\right)$

#2. Heat equation, $k=1$, $L=\pi$, $u(x,t)=\sum_{n=1}^{\infty} c_n e^{-n^2 t}\sin(nx)$, $u(x,0)=\sum_{n=1}^{\infty} c_n \sin(nx)=2$

$\lambda_n=\left(\frac{n\pi}{\pi}\right)^2=n^2$, $n=1,2,\ldots$

$X_n=\sin(nx)$

Use Fourier sine series $c_n=\frac{2}{\pi}\int_0^{\pi} 2\cdot\sin(nx)\,dx=\frac{4}{\pi}\left(\frac{-\cos(nx)}{n}\right)\Big|_0^{\pi}$

$$c_n=\frac{-4}{n\pi}\left(\cos(n\pi)-1\right)=\frac{4}{n\pi}\left(1-(-1)^n\right)$$

$$\boxed{u(x,t)=\sum_{n=1}^{\infty}\frac{4}{n\pi}\left(1-(-1)^n\right)e^{-n^2 t}\sin(nx)}$$