Math 4350.004 Spring 2018
3.7 Infinite Series

1. Let $X := (x_n)$ be a sequence in $\mathbb{R}$. The **infinite series generated by** X is the **sequence of partial sums** $S := (s_k)$ defined by

$$\begin{aligned}
s_1 &:= x_1 \\
s_2 &:= x_1 + x_2 = s_1 + x_2 \\
s_3 &:= x_1 + x_2 + x_3 = s_2 + x_3 \\
&\cdots \\
s_k &:= x_1 + x_2 + \cdots + x_k = s_{k-1} + x_k
\end{aligned}$$

The **infinite series** is expressed as $\sum(x_n) = \sum x_n = \sum_{n=1}^{\infty} x_n$. The infinite series does not have to begin at $n = 1$. To show an infinite series converges, show that the sequence of partial sums (s_k) converges. That is, $\lim s_k = L$ for some real number L or show (s_k) is a Cauchy sequence: let $\epsilon > 0$ and there exists $K \in \mathbb{N}$ such that $|s_m - s_n| < \epsilon$ for $m, n \geq K$.

2. The **geometric series** converges for $|r| < 1$:

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \cdots + r^n + \cdots = \frac{1}{1-r}$$

Does the series $\sum_{n=2}^{\infty} \frac{3}{2^n}$ converge? If yes, find the limit of the series.

3. The **harmonic series** $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and the **alternating harmonic series** $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.

4. The **p-series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $0 < p \leq 1$.

Do the following series converge or diverge? $\sum_{n=2}^{\infty} \frac{\pi}{\sqrt{n}}$, $\sum_{n=5}^{\infty} \frac{n+1}{n^2}$

5. **The nth Term Test**: If the series $\sum x_n$ converges, then $\lim(x_n) = 0$.
State the contrapositive of the nth Term Test.
Show that the series $\sum \cos(n)$ diverges.
Show that the series $\sum \cos(n)/n^2$ converges.
Give a counterexample to show that the converse of the nth Term Test is false.

6. **Comparison Test**: Let $X := (x_n)$ and $Y := (y_n)$ be real sequences and for some natural number $K \in \mathbb{N}$, $0 \leq x_n \leq y_n$ for $n \geq K$.
(a) If $\sum y_n$ converges, then $\sum x_n$ converges.
(b) If $\sum x_n$ diverges, then $\sum y_n$ diverges.

7. True or False? If it is true, give a proof. If it is false, give a counterexample.

   (a) If $\sum x_n$ and $\sum y_n$ are both convergent, then $\sum(x_n + y_n)$ is convergent.
   (b) If $\sum x_n$ and $\sum y_n$ are both divergent, then $\sum(x_n + y_n)$ is divergent.
   (c) If $\sum x_n$ is convergent and $\sum y_n$ is divergent, then $\sum(x_n + y_n)$ is divergent.
   (d) If $\sum x_n$ with $x_n > 0$ is convergent, then $\sum x_n^2$ is convergent.