Review for Exam # 2, Math 2450, Fall 2015
Chapter Sections: 11.4-11.8, 12.1-12.5,12.7-12.8, 13.2

Review: Four Homework Assignments, Webwork 5, 6, 7 and Quizzes # 2,3.

Tangent plane to $z = f(x, y)$ at a point $P_0(x_0, y_0, z_0)$: $\vec{N} =< f_x(x_0, y_0), f_y(x_0, y_0), -1 >=< A, B, -1 >$ is normal to the surface. Tangent plane is $z - z_0 = A(x - x_0) + B(y - y_0)$.
Tangent plane to an implicit surface $F(x, y, z) = c$ at a point $P_0 = P(x_0, y_0, z_0)$: $\nabla F = \langle F_x, F_y, F_z\rangle|_{P_0} =< A, B, C >$ evaluated at P_0 is the is normal to the surface, so the tangent plane is $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$. The line normal to the tangent plane at P_0: $x = x_0 + At,\ y = y_0 + Bt,\ z = z_0 + Ct$.

Incremental Approximation at $P(x_0, y_0)$: $\Delta f = f(x + \Delta x, y + \Delta y) - f(x, y) \approx f_x\Delta x + f_y\Delta y$, where the partials are evaluated at (x_0, y_0); Total differential for $f(x, y, z)$: $df = f_xdx + f_ydy + f_zdz$.
Chain rule for $z = f(x, y)$, $x \equiv x(t)$, $y \equiv y(t)$ $z \equiv f(x(t), y(t))$: $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt}$. For $z = f(x, y)$, $x \equiv x(u, v)$, $y \equiv y(u, v)$, $z = f(x(u, v), y(u, v))$:

$$\begin{aligned}\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.\\ \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\end{aligned}$$

Implicit differentiation $F(x, y(x)) = c : F_x + F_y\dfrac{dy}{dx} = 0$ which means $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y}$.
Gradient of $f(x, y, z)$ is the vector $\nabla f = f_x\vec{i} + f_y\vec{j} + f_z\vec{k}$. The directional derivative $D_{\vec{u}}f = \nabla f \cdot \vec{u}$ is a scalar representing the slope of $f(x, y, z)$ in the direction $\vec{u}$, where $\vec{u}$ is a unit vector. Maximal property: largest value of the directional derivative $D_{\vec{u}}f$ at the point $P_0(x_0, y_0, z_0)$ occurs in the direction of the gradient, ∇f_0, and smallest value occurs in the negative direction of the gradient, $-\nabla f_0$. The largest value of the directional derivative at the point P_0 is the magnitude of the gradient $\|\nabla f_0\|$ and smallest value is negative of the magnitude of the gradient $-\|\nabla f_0\|$.

Local extrema of a function of two variables $z = f(x, y)$: find the critical points P_0 in the domain by solving simultaneously the equations $f_x = 0$ and $f_y = 0$ for (x, y). Test the critical points using the Second partial derivative test where $D = f_{xx}f_{yy} - f_{xy}^2$ is evaluated at a critical point P_0. There is a *relative maximum* $f(P_0)$ if at P_0, $D > 0$ and $f_{xx} < 0$, a *relative minimum* if at P_0, $D > 0$ and $f_{xx} > 0$ and a *saddle point* if at P_0, $D < 0$. If $D = 0$, test fails.
Absolute extrema of a function of two variables $z = f(x, y)$ on a closed and bounded set S: find the critical points P_0 in S and on the boundary of S, then evaluate the function at these critical points and boundary points. The largest f value is the absolute maximum and the smallest is the absolute minimum.

Lagrange Multipliers is a method to find a maximum or minimum of a function subject to one or more constraints. Maximize or minimize $f(x, y, z)$ subject to $g(x, y, z) = c$ =constant, then solve $f_x = \lambda g_x$, $f_y = \lambda g_y$, $f_z = \lambda g_z$ and $g(x, y, z) = c$ for (x, y, z, λ) and check which point gives the largest (or smallest) f value.

Multiple integration over rectangular regions: $R:\ a \le x \le b,\ c \le y \le d$. Fubini's Theorem: If $f(x, y)$ is continuous on R, then the double integral can be written as an iterated integral:

$$\iint_R f(x, y)dA = \int_c^d \int_a^b f(x, y)dx\,dy = \int_a^b \int_c^d f(x, y)dy\,dx.$$

Multiple integration over nonrectangular region R: $a \le x \le b$ and $g(x) \le y \le h(x)$.

$$\iint_R f(x, y)dA = \int_a^b \int_{g(x)}^{h(x)} f(x, y)\,dy\,dx$$

Change the order of integration to $dx\,dy$.
Polar Coordinates (r, θ): r is the distance in xy plane from the origin, $r \ge 0$, θ is the polar angle measured from the positive x axis counterclockwise, $0 \le \theta \le 2\pi$. $x = r\cos(\theta)$, $y = r\sin(\theta)$, $r^2 = x^2 + y^2$, $\tan(\theta) = \dfrac{y}{x}$, $dA = r\,dr\,d\theta$.

$$\iint_R f(x, y)\,dx\,dy = \iint_{R^*} f(r\cos(\theta), r\sin(\theta))\,rdr\,d\theta.$$

Surface area of $z = f(x,y)$ inside the region R in a plane: $\iint_R \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\, dx\, dy$.

Triple integrals:

$$\iiint_D f(x,y,z) dV = \iiint_D f(x,y,z)\, dx\, dy\, dz.$$

If $f(x,y,z) = 1$, the triple integral is the volume of D. Change order of integration to $dy\, dx\, dz$ or $dz\, dy\, dx$, etc.

Cylindrical Coordinates: Coordinates on a cylinder (r, θ, z), polar coordinates in x and y but z remains the same. $x = r\cos(\theta),\ y = r\sin(\theta),\ z = z,\ dV = dz\, r dr d\theta$.

$$\iiint_D f(x,y,z)\, dx\, dy\, dz = \iiint_{D^*} f(r\cos(\theta), r\sin(\theta), z)\, r\, dz\, dr\, d\theta$$

Spherical Coordinates: Coordinates on a sphere (ρ, θ, ϕ): ρ is distance from origin, $\rho \geq 0$, θ is polar angle measured in xy plane from the positive x axis counterclockwise, $0 \leq \theta \leq 2\pi$ and ϕ is the angle measured from the north pole (positive z axis) toward the south pole (negative z axis), $0 \leq \phi \leq \pi$.

$x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ z = \rho\cos\phi,\ \ \rho^2 = x^2 + y^2 + z^2,\ \ dV = \rho^2\sin(\phi) d\rho d\phi d\theta$

$$\iiint_D f(x,y,z)\, dx\, dy\, dz = \iiint_{D*} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2\sin(\phi)\, d\rho\, d\phi\, d\theta.$$

Jacobians: Change of Variables: $x = g(u,v)$ and $y = h(u,v)$. Jacobian determinant:

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

$$\iint_D f(x,y)\, dy\, dx = \iint_{D^*} f(g(u,v), h(u,v))|J(u,v)|\, du\, dv$$

Vector Field: $\vec{F}(x,y,z) =< M, N, P >$ $\vec{R}(t) =< x(t), y(t), z(t) >$ Use parametric equations for $x(t), y(t)$, and $z(t)$ to define the curve C, $a \leq t \leq b$.

Line integral: $\int_C f(x,y,z) ds = \int_a^b f(x(t), y(t), z(t))\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt$.

Line integral: $\int_C [f(x,y)dx + g(x,y)dy] = \int_a^b [f(x(t), y(t))x'(t)dt + g(x(t), y(t))y'(t)\, dt]$.

Line integral: $\int_C \vec{F} \cdot d\vec{R} = \int_C [M dx + N dy + P dz]$.

Several curves: $C = C_1 \cup C_2 \cup C_3$: $\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3}$.

Review Exercises:

Chapter 11, pp. 910-11 # 29,35,37,43,47,49,51,55,63,65;

Chapter 12, pp. 1010-13 #5,9,17,19,21,23,29,35,37,39,40,42,53,61;

Chapter 13, pp. 1098-99 # 10,23.