Math 2450.010, Quiz 2
Fall 2015

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No calculators. Turn off all electronic devices.

Name Answers

1. (10 pts) Suppose $z = f(x, y) = (x + \ln(y))^2$ and $x = v^2 - u^2$, $y = e^{u^2}$.

   (a) Use the chain rule to compute the partial derivatives. Express your answer in terms of u and v and simplify.

   $$\frac{\partial z}{\partial u} = 2(x+\ln y)\cdot(-2u) + 2(x+\ln(y))\frac{1}{y}2ue^{u^2} = -4u(x+\ln(y)) + 4u(x+\ln y)\frac{e^{u^2}}{y}$$

   $$= 4u(x+\ln(y))(-1+1) = 0$$

   $$\frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u}$$

   $$\frac{\partial z}{\partial v} = 2(x+\ln y)\cdot(2v) + 2(x+\ln(y))\frac{1}{y}\cdot 0 = 4v(v^2-u^2-\ln e^{u^2}) = 4v\cdot v^2 = 4v^3$$

   $$\frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

   or $z = (v^2-u^2+\ln e^{u^2})^2 = (v^2-u^2+u^2)^2 = (v^2)^2 = v^4$

   $\frac{\partial z}{\partial u} = 0$ $\quad \frac{\partial z}{\partial v} = 4v^3$

2. (10 pts) Write an equation for a <u>plane</u> that is tangent to the following surface: $z-5 = 2(x-1)+8(y-3)$

   (a) $z = x^2 + 4(y - 2)^2$ at the point $P_0(1, 3, 5)$.

   $\vec{N} = \langle f_x, f_y, -1\rangle\Big|_{P_0} = \langle 2x, 8(y-2), -1\rangle\Big|_{P_0} = \langle 2, 8, -1\rangle$ $\quad 2x+8y-z-21=0$

   $2x+8y-z+D=0 \Rightarrow 2+24-5+D=0 \Rightarrow D=-21$

   (b) $x^2 + 2xy + z^2 = 4$ at the point $P_0(1, 1, 1)$.

   $\vec{N} = \langle F_x, F_y, F_z\rangle\Big|_{P_0} = \langle 2x+2y, 2x, 2z\rangle\Big|_{P_0} = \langle 4, 2, 2\rangle$ $\quad 2x+y+z-4=0$

   $4x+2y+2z+D=0 \Rightarrow 4+2+2+D=0 \Rightarrow D=-8 \Rightarrow 4x+2y+2z-8=0$

3. (5 pts) Given the function $z = f(x, y) = 3x^2y + e^{2y} - 5$, compute the following:

   (a) Differential of f, $df =$ $(6xy)dx + (3x^2+2e^{2y})dy$.

   (b) Approximate the increment $\Delta f(1.99, 0.01)$ using the differential of f.

   $\Delta f \approx df = 6(2)(0)(-.01) + (3\cdot 4+2)(0.01) = 0.14$

   at $(x_0,y_0)=(2,0)$ $\quad dx=\Delta x = -0.01$ $\quad dy = \Delta y = 0.01$

   $f(1.99, 0.01) \approx f(2,0) + df$

4. (10 pts) Given the function $z = f(x, y) = x^2y^2 - \sin(x-1)$, compute the following:

(a) Gradient of f, $\nabla f(x, y)$ at $P_0(1,1,1)$.

$$\nabla f|_{P_0} = (2xy^2 - \cos(x-1))\vec{i} + 2x^2y\vec{j}\Big|_{P_0} = (2-1)\vec{i} + 2\vec{j} = \vec{i} + 2\vec{j}$$

(b) Directional derivative $D_{\vec{u}}f(x, y)$ at the point $P_0(1,1,1)$ in the direction of $\vec{v} = \vec{i} - 3\vec{j}$.

$$\vec{u} = \frac{\vec{i}}{\sqrt{10}} - \frac{3}{\sqrt{10}}\vec{j} \qquad D_{\vec{u}}f(1,1) = \nabla f(1,1)\cdot\vec{u} = \langle 1, 2\rangle \cdot \langle \tfrac{1}{\sqrt{10}}, \tfrac{-3}{\sqrt{10}}\rangle$$
$$= \frac{1}{\sqrt{10}} - \frac{6}{\sqrt{10}} = \frac{-5}{\sqrt{10}}$$

(c) Direction $\vec{u}$ from $P_0(1,1,1)$ in which f increases most rapidly and the magnitude of the greatest rate of increase.

$\vec{u} = \dfrac{\nabla f}{\|\nabla f\|}$ and greatest magnitude is $\|\nabla f\|$

$$\vec{u} = \frac{\vec{i}}{\sqrt{5}} + \frac{2}{\sqrt{5}}\vec{j} \qquad \|\nabla f\| = \sqrt{5} \qquad D_{\vec{u}}f = \|\nabla f\|\cos\theta$$

5. (15 pts) Consider the function $z = f(x, y) = x^3 + y^3 - 3xy$.

(a) Find all critical points.

(b) Apply the second derivative test for a function of two variables to classify each critical point as a relative minimum, relative maximum, or saddle point.

$$\left.\begin{aligned} f_x &= 3x^2 - 3y = 0 \Rightarrow y = x^2 \\ f_y &= 3y^2 - 3x = 0 \Rightarrow x = y^2 \end{aligned}\right\} \Rightarrow y = (y^2)^2 = y^4 \Rightarrow y^4 - y = 0$$
$$y(y^3 - 1) = 0$$
$$\boxed{y = 0, y = 1}$$

$y = 0,\ x = 0$ 2 critical points: (0,0) (1,1)

$y = 1 \quad x = 1$

$$D(x,y) = f_{xx}f_{yy} - (f_{xy})^2 = (6x)(6y) - (-3)^2$$

(0,0) $D(0,0) = -9 < 0 \Rightarrow$ at (0,0) saddle point

(1,1) $D(1,1) = 36 - 9 > 0$ and $f_{xx}(1,1) = 6 > 0$ ∪

$\Rightarrow$ at (1,1) relative minimum

See graph of $z = f(x,y)$

$plot3d(x^3 + y^3 - 3\cdot x\cdot y, x = -1..2, y = -1..2);$