Honors Math 2450, Quiz 2
Fall 2015

Show all of your work. Circle your answers. Name Answers
No calculators. Turn off all electronic devices.

1. (10 pts) Suppose $z = f(x,y) = (e^y - x)^2$ and $x = v^2 - u^2$, $y = \ln(u^2 + v^2)$.

   (a) Use the chain rule to compute the partial derivatives. Express your answer in terms of u and v and simplify.

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = 2(e^y - x)(-1)(-2u) + 2(e^y - x)(e^y)\frac{2u}{u^2+v^2}$$

$$= (u^2+v^2-v^2+u^2)4u + 4u\frac{(u^2+v^2-v^2+u^2)\cancel{(u^2+v^2)}}{\cancel{u^2+v^2}}$$

$$= 4u(2u^2) + 4u(2u^2) = 16u^3$$

$$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = 2(e^y - x)(-1)2v + 2(e^y - x)(e^y)\frac{2v}{u^2+v^2}$$

$$= -4v(2u^2) + 4v(2u^2) = 0$$

OR

$$z = (2u^2)^2 = 4u^4 \qquad \frac{\partial z}{\partial u} = 16u^3 \qquad \frac{\partial z}{\partial v} = 0$$

2. (5 pts) Given the function $z = f(x,y) = 3x^2y + e^{xy} - 5$, compute the following:

   (a) Differential of f, $df = (6xy + ye^{xy})dx + (3x^2 + xe^{xy})dy$.

   (b) Approximate $f(2.99, 0.01)$ using the differential of f when $(x_0, y_0) = (3, 0)$

$$f(2.99, 0.01) \approx f(3,0) + df = \underbrace{(27(0)+1-5)}_{f(3,0)} + \underbrace{0(-0.1) + (27+3)(0.01)}_{df}$$

$$= \underbrace{-4}_{f(3,0)} + \underbrace{0.30}_{df} = -3.70$$

3. (10 pts) Write an equation for a plane in standard form that is tangent to the following surface:

   (a) $z = (x+1)^2 + 4(y-2)^2$ at the point $P_0(1,3,8)$.

$$f_x = 2(x+1) = 4 \qquad z - 8 = 4(x-1) + 8(y-3)$$
$$f_y = 8(y-2) = 8$$
$$4x + 8y - z - 20 = 0$$

   (b) $x^3 + 2xy + z^3 = 4$ at the point $P_0(1,1,1)$.

$$f_x = 3x^2 + 2y = 5 \qquad 5x + 2y + 3z + D = 0$$
$$f_y = 2x = 2 \qquad 5x + 2y + 3z - 10 = 0$$
$$f_z = 3z^2 = 3$$

4. (10 pts) Given the function $z = f(x,y) = x^2y^2 + 3\sin(x-1)$, compute the following:

(a) Gradient of f, $\nabla f(x,y)$ at $P_0(1,1,1)$.

$$\nabla f = [2xy^2 + 3\cos(x-1)]\vec{i} + 2x^2y\vec{j}$$

$$\nabla f|_{P_0} = 5\vec{i} + 2\vec{j}$$

(b) Directional derivative $D_{\vec{u}}f(x,y)$ ~~at the point~~ $P_0(1,1,1)$ in the direction of $\vec{v} = 2\vec{i} - 3\vec{j}$.

$$D_{\vec{u}}f(x,y) = \nabla f \cdot \vec{u} = \langle 5, 2\rangle \cdot \left\langle \frac{2}{\sqrt{13}}, \frac{-3}{\sqrt{13}} \right\rangle = \frac{10}{\sqrt{13}} - \frac{6}{\sqrt{13}}$$

$$= \frac{4}{\sqrt{13}}$$

(c) Direction $\vec{u}$ from $P_0(1,1,1)$ in which f increases most rapidly and the magnitude of the greatest rate of increase.

Direction $\vec{u} = \dfrac{\nabla f}{\|\nabla f\|} = \dfrac{5\vec{i} + 2\vec{j}}{\sqrt{29}} = \dfrac{5}{\sqrt{29}}\vec{i} + \dfrac{2}{\sqrt{29}}\vec{j}$

Magnitude $\|\nabla f\| = \sqrt{29}$

5. (15 pts) Consider the function $z = f(x,y) = x^3 + y^3 - 6xy$.

(a) Find all critical points.

(b) Apply the second derivative test for a function of two variables to classify each critical point as a relative minimum, relative maximum, or saddle point.

$$f_x = 3x^2 - 6y \qquad x^2 = 2y \qquad y = \frac{x^2}{2}$$

$$f_y = 3y^2 - 6x \qquad y^2 = 2x \qquad x = \frac{y^2}{2}$$

$$\Rightarrow x = \frac{\left(\frac{x^2}{2}\right)^2}{2}$$

$$\Rightarrow x = \frac{x^4}{8} \Rightarrow \quad x^4 - 8x = 0 \qquad x = 0, x = 2$$

$$x(x^3 - 8) = 0 \qquad y = 0, y = 2$$

Critical Points $(0,0)$ $(2,2)$

Test $D(x,y) = f_{xx}f_{yy} - (f_{xy})^2$

$$= (6x)(6y) - (-6)^2$$

$D(0,0) = -36 \Rightarrow$ Saddle point at $(0,0)$

$D(2,2) = 36(4) - 36 > 0$ and $f_{xx}(2,2) = 12 > 0$ ∪

$\Rightarrow$ Relative minimum at $(2,2)$

see graph of $z = f(x,y)$

$plot3d(x^3 + y^3 - 6 \cdot x \cdot y, x = -2..4, y = -2..4);$