Math 2450.H01, Fall 2015
Exam # 2

Name Answers

Show all of your work. Do not use calculators.
Turn off all electronic devices. In Problems 1-3, show work and circle the correct answer. (Total=125 pts)

1. (10 pts) Find the equation of the plane tangent to the surface $F(x, y, z) = x^3 + 2yz - z^2 = 0$ at the point $(2, 1, -2)$.

(a) $6(x-2)+2(y-1)+3(z+2)=0$
(b) ~~$3x^2(x-2)+2z(y-1)+2(y-z)(z+2)=0$~~
(c) $2x+y-2z=9$
(d) $6(x-2)-2(y-1)+3(z+2)=0$
(e) $\langle 12, -4, 6\rangle$
(f) $12x-4y+6z=5$

$\nabla F|_P = \langle 3x^2, 2z, 2y-2z\rangle|_P = \langle 12, -4, 6\rangle$ $\quad 12x-4y+6z=D$ $\quad$ or $12(x-2)-4(y-1)+6(z+2)=0$

$24-4-12=8$

$6x-2y+3z=4$ $\quad \boxed{6(x-2)-2(y-1)+3(z+2)=0}$

2. (10 pts) Use Lagrange multipliers to find the minimum of the function $f(x, y, z) = x^2 + 3y^2 + 4z^2$ subject to the constraint $x + 6y - 4z = 17$. The minimum occurs at the point (x_0, y_0, z_0) given by

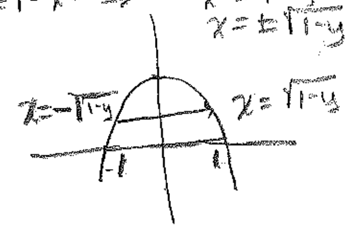
(a) $(2, 2, -1/4)$
(b) $(1, 1, -1)$
(c) $(3, 1, -2)$
(d) $(1, 3, 1/2)$
(e) $(0, 0, 0)$
(f) $(1, 2, -1)$

$2x = \lambda$
$6y = 6\lambda$
$8z = -4\lambda$

$\lambda = 2x = -2z = y$
$x + 12x + 4x = 17$
$17x = 17$
$x = 1$
$y = 2$
$z = -1$

3. (10 pts) Change the order of integration $I = \int_{-1}^{1}\int_{0}^{1-x^2} f(x, y)\, dy\, dx$ after sketching the region of integration in the x-y plane.

$y = 1 - x^2 \Rightarrow x^2 = 1-y$
$x = \pm\sqrt{1-y}$



(a) $\int_0^{1-x^2}\int_{-1}^{1} f(x, y)\, dx\, dy$
(b) $\int_0^1\int_{-\sqrt{y}}^{\sqrt{y}} f(x, y)\, dx\, dy$
(c) $\int_{-1}^{1}\int_{\sqrt{y}}^{2} f(x, y)\, dx\, dy$
(d) $\int_0^1\int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x, y)\, dx\, dy$
(e) $\int_{-1}^{1}\int_{0}^{\sqrt{1-y}} f(x, y)\, dx\, dy$

4. (10 pts) Use the chain rule to compute the partial derivatives if $z = \ln(u^3 + 3uv^3)$ and $u = 4x^2y$, and $v = e^{xy}$. Leave your answer in terms of u, v, x and y.

(a) $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \boxed{\frac{3u^2+3v^3}{u^3+3uv^3}\cdot 8xy + \frac{9uv^2}{u^3+3uv^3}\cdot ye^{xy}}$

(b) $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \boxed{\frac{3u^2+3v^3}{u^3+3uv^3}\cdot 4x^2 + \frac{9uv^2}{u^3+3uv^3}\cdot xe^{xy}}$

5. (10 pts) Given the function $f(x, y, z) = x^2z + x\ln(z) - \sin(2y)$ and the point $P(1, \pi/2, 1)$, compute

(a) the directional derivative $D_{\vec{u}}f(P)$ in the direction of the vector $\vec{v} = 3\vec{i} - \vec{j} + 2\vec{k}$.

(b) the smallest value of the directional derivative at P.

(a) $D_{\vec{u}}f(P) = \nabla f(P) \cdot \vec{u} = \left.\underbrace{\langle 2xz + \ln z, -2\cos(2y), x^2 + \frac{x}{z}\rangle}_{\langle 2+\ln 1,\ 2,\ 2\rangle}\right|_P \cdot \vec{u}$

$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{3\vec{i} - \vec{j} + 2\vec{k}}{\sqrt{14}}$

$D_{\vec{u}}f(P) = \langle 2,2,2\rangle \cdot \langle \frac{3}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{2}{\sqrt{14}}\rangle = \frac{6-2+4}{\sqrt{14}} = \boxed{\frac{8}{\sqrt{14}}}$

(b) $-\|\nabla f(P)\| = -\|\langle 2,2,2\rangle\| = -\sqrt{4+4+4} = -\sqrt{12} = \boxed{-2\sqrt{3}}$

6. (15 pts) Given the function $f(x, y) = x^3 + xy - x - y^2$ find all critical points. Then use the second partials test to classify each critical point as a relative maximum, a relative minimum or a saddle point.

$f_x = 3x^2 + y - 1 = 0$

$f_y = x - 2y = 0$

$\Rightarrow x = 2y$

$3(2y)^2 + y - 1 = 0 \Rightarrow 12y^2 + y - 1 = 0$

$(3y+1)(4y-1)$

$\boxed{y = -\frac{1}{3}, y = \frac{1}{4}}$

Critical points: $(-\frac{2}{3}, -\frac{1}{3})$ $(\frac{1}{2}, \frac{1}{4})$

Discriminant $D = f_{xx}f_{yy} - (f_{xy})^2$

$= (6x)(-2) - (1)^2$

$= -12x - 1$

$\boxed{(-\frac{2}{3}, -\frac{1}{3})}$: $D = -12(-\frac{2}{3}) - 1 = 8 - 1 > 0$ $f_{xx} = -4 < 0$

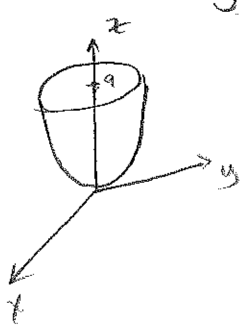
$\Rightarrow$ Relative Max

$\boxed{(\frac{1}{2}, \frac{1}{4})}$: $D = -12(\frac{1}{2}) - 1 < 0$

$\Rightarrow$ Saddle point

7. (30 pts) Use polar or cylindrical coordinates.

(a) Find the surface area of the portion of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 9$.



$$S = \iint_R \sqrt{(f_x)^2 + (f_y)^2 + 1}\, dA$$

$\partial R: x^2+y^2=9 \Rightarrow$ circle rad $=3$

$z = f(x,y) = x^2+y^2$

$f_x = 2x, \; f_y = 2y$

$$\iint \sqrt{\underbrace{(2x)^2+(2y)^2+1}_{4(x^2+y^2)+1}}\, dA$$

$$\int_0^{2\pi}\int_0^3 \underbrace{\sqrt{4r^2+1}\; r\,dr\,d\theta}_{\text{Substitution}} = \int_0^{2\pi} \frac{1}{8}\int_0^3 (4r^2+1)^{\frac{1}{2}} 8r\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{8}(4r^2+1)^{3/2}\cdot\frac{2}{3}\Big|_{r=0}^{r=3} d\theta = \int_0^{2\pi}\left(\frac{1}{12}(37)^{3/2} - \frac{1}{12}\right)d\theta = \boxed{\frac{\pi}{6}(37^{3/2}-1)}$$

(b) Use a triple integral to find the volume between between the paraboloid $z = 4 - x^2 - y^2$ and the x-y plane.

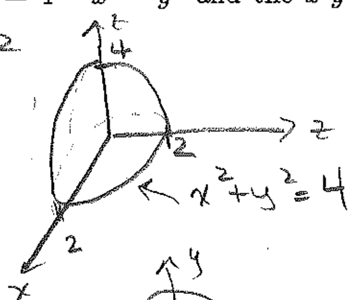
$$\int_0^{2\pi}\int_0^2\int_0^{4-r^2} 1 \cdot dz\, r\,dr\,d\theta$$

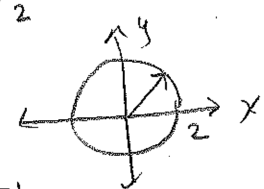
$0 \le z \le 4 - x^2 - y^2$

$0 \le z \le 4 - r^2$

$0 \le r \le 2$

$0 \le \theta \le 2\pi$



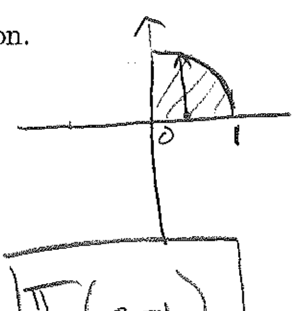


$$= \int_0^{2\pi}\int_0^2 \underbrace{(4-r^2)\,r}_{4r-r^3}\,dr\,d\theta$$

$$= \int_0^{2\pi} 2r^2 - \frac{r^4}{4}\Big|_{r=0}^{r=2} d\theta = \int_0^{2\pi}(8-4)\,d\theta = \boxed{8\pi}$$

(c) Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} e^{x^2+y^2}\, dy\, dx$ after sketching the region of integration.

$y = \sqrt{1-x^2}$



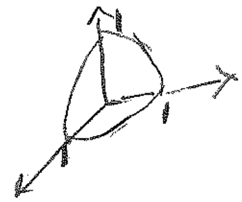
$$\int_0^1\int_0^{\sqrt{1-x^2}} e^{x^2+y^2}\,dy\,dx = \int_0^{\frac{\pi}{2}}\int_0^1 \underbrace{e^{r^2} r\,dr}_{\text{Substitution}}\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}e^{r^2}\Big|_0^1 d\theta = \int_0^{\frac{\pi}{2}}\left(\frac{1}{2}e - \frac{1}{2}\right)d\theta = \boxed{\frac{\pi}{4}(e-1)}$$

8. (20 pts) Evaluate the integrals.

(a) $\iiint_D \underbrace{\sqrt{x^2+y^2+z^2}}_{\sqrt{\rho^2}=\rho}\,dV$, where D is the portion of the solid sphere $x^2+y^2+z^2 \le 1$ that lies in the <u>first octant.</u>

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \underbrace{\rho\cdot\rho^2\sin\phi}_{\rho^3\sin\phi}\,d\rho\,d\phi\,d\theta = \int_0^{\pi/2}\int_0^{\pi/2} \frac{\rho^4}{4}\Big|_{\rho=0}^{\rho=1}\sin\phi\,d\phi\,d\theta$$



$$\int_0^{\pi/2}\frac{1}{4}(-\cos\phi)\Big|_0^{\pi/2}\,d\theta = \int_0^{\pi/2}\frac{1}{4}\,d\theta = \boxed{\frac{\pi}{8}}$$

(b) $\iint_D (x-5y)^5(2x+y)^3\,dy\,dx$, where D is the region bounded by $x-5y=0$, $x-5y=2$, $2x+y=0$ and $2x+y=1$. Let $u=x-5y$ and $v=2x+y$. Find the Jacobian of the transformation $J(u,v)$ and evaluate the integral.

$$\iint u^5v^3\frac{1}{11}\,du\,dv$$

$$J(u,v)=\frac{\partial(x,y)}{\partial(u,v)} \qquad \frac{1}{J(x,y)} = \frac{1}{11}$$

$$J(x,y)=\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix}1 & -5\\ 2 & 1\end{vmatrix} = 1+10 = 11$$

limits: $x-5y=0 \Rightarrow u=0$ $\quad 2x+y=0 \Rightarrow v=0$
$x-5y=2 \Rightarrow u=2$ $\quad 2x+y=1 \Rightarrow v=1$

$$\frac{1}{11}\int_0^1\int_0^2 u^5v^3\,du\,dv = \frac{1}{11}\frac{u^6}{6}\Big|_0^2\cdot\frac{v^4}{4}\Big|_0^1 = \frac{2^6}{6\cdot 11}\cdot\frac{1}{4}$$
$$=\boxed{\frac{8}{33}}$$

9. (10 pts) Compute the line integral $\int_C \vec{F}\cdot d\vec{R}$, where C is the curve $y=x^2$ from (0,0) to (2,4) and $\vec{F}=y^2\vec{i}+xy\vec{j}$.

$$\int_C \langle y^2, xy\rangle\cdot\langle dx, dy\rangle = \int_C y^2\,dx + xy\,dy$$ – replace y by x^2, $x=0$ to $x=2$

$$= \int_0^2 (x^2)^2\,dx + x(x^2)\cdot 2x\,dx$$

$$= \int_0^2 (x^4+2x^4)\,dx = \int_0^2 3x^4\,dx = \frac{3}{5}x^5\Big|_0^2$$

$$= \frac{3}{5}(32) = \boxed{\frac{96}{5}}$$