Math 2450.010, Fall 2015

Exam # 2

Name Answers

Show all of your work. Do not use calculators.

Turn off all electronic devices. In Problems 1-4, show work and circle the correct answer. (Total=115 pts)

1. (10 pts) Find the equation of the plane tangent to the surface $F(x,y,z) = x^2 + 2yx - z^2 = 5$ at the point $(1,3,-2)$.

(a) $(x-1) + 2(y-3) + 4(z+2) = 0$
(b) $(2x+2y)(x-1) + 2x(y-3) - 2z(z+2) = 0$
(c) $4x + y + 2z = 3$
(d) $4(x-1) + (y-3) - 2(z+2) = 0$
(e) $\langle 8, 2, 4\rangle$
(f) None of the answers.

$\vec{N}\Big|_{P(1,3,-2)} = \langle 2x+2y, 2x, -2z\rangle\Big|_{P(1,3,-2)} = \langle 8,2,4\rangle \Rightarrow$ Tangent Plane $8x+2y+4z=D \Rightarrow 8(1)+2(3)+4(-2)=6$

$\Rightarrow 8x+2y+4z=6 \Rightarrow \boxed{4x+y+2z=3}$

2. (10 pts) Evaluate the triple integral $I = \int_0^1 \int_0^x \int_1^y 2z\, dz\, dy\, dx$

(a) $I = \frac{5}{12}$
(b) $I = -\frac{5}{12}$
(c) $I = 4$
(d) $I = (y^2-1)x$
(e) $I = -\frac{11}{12}$
(f) $I = \frac{11}{12}$

$\int_0^1\int_0^x z^2\Big|_{z=1}^{z=y} dy\,dx = \int_0^1\int_0^x (y^2-1)\,dy\,dx$

$= \int_0^1 \left(\frac{y^3}{3} - y\right)\Big|_{y=0}^{y=x} dx = \int_0^1 \left(\frac{x^3}{3} - x\right)dx$

$= \frac{x^4}{12} - \frac{x^2}{2}\Big|_0^1 = \frac{1}{12} - \frac{1}{2} = \boxed{-\frac{5}{12}}$

3. (10 pts) Application of Lagrange multipliers to find the minimum of the function $f(x,y,z) = x^2 + y^2 + 2z^2$ subject to the constraint $x + 3y + 2z = 6$ shows that the minimum is at the point (x_0, y_0, z_0) given by

(a) $(1,3,2)$
(b) $(1,1,1)$
(c) $(1/2, 3/2, 1/2)$
(d) $(0,0,0)$
(e) $(2,1,1/2)$
(f) $(4, 1/3, 1/2)$

Lagrange Multipliers $\left.\begin{array}{l} f_x=\lambda g_x \\ f_y=\lambda g_y \\ f_z=\lambda g_z \end{array}\right\} \Rightarrow \begin{array}{l} 2x=\lambda \\ 2y=3\lambda \\ 4z=2\lambda \end{array}$

Thus, $\lambda = 2x = \frac{2}{3}y = 2z$

$x=z \quad 3x=y \Rightarrow x+3(3x)+2x=6$

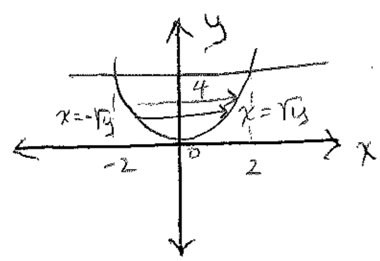
$12x=6$

$x=\frac{1}{2}$

$\boxed{x=\frac{1}{2},\ z=\frac{1}{2}\ y=\frac{3}{2}}$

4. (10 pts) Change the order of integration $I = \int_{-2}^{2}\int_{x^2}^{4} f(x,y)\, dy\, dx$ after sketching the region of integration in the x-y plane.

(a) $\int_{x^2}^{4}\int_{-2}^{2} f(x,y)\, dx\, dy$
(b) $\int_0^4 \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)\, dx\, dy$
(c) $\int_0^4 \int_{\sqrt{y}}^{2} f(x,y)\, dx\, dy$
(d) $\int_{-4}^{4}\int_{\sqrt{y}}^{2} f(x,y)\, dx\, dy$
(e) $\int_0^2 \int_0^4 f(x,y)\, dx\, dy$



5. (10 pts) Use the chain rule to compute the partial derivatives if $z = \ln(u^2 + v^2)$ and $u = 3x - 9y$, and $v = x^2 e^{3y}$. Leave your answer in terms of u, v, x and y.

(a) $\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial x} = \dfrac{2u}{u^2+v^2}\cdot 3 + \dfrac{2v}{u^2+v^2}\cdot 2xe^{3y} = \boxed{\dfrac{6u+4xe^{3y}}{u^2+v^2}}$

(b) $\dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial y} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial y} = \dfrac{2u}{u^2+v^2}\cdot(-9) + \dfrac{2v}{u^2+v^2}\cdot 3x^2e^{3y} = \boxed{\dfrac{-18u+6vx^2e^{3y}}{u^2+v^2}}$

6. (10 pts) Given the function $z = f(x, y) = x^2 + \sin(2y + 4x)$ and the point $P(0, \pi/2)$, compute

(a) the directional derivative $D_{\vec{u}} f(P)$ in the direction of the vector $\vec{v} = 2\vec{\imath} - 5\vec{\jmath}$.

(b) the largest value of the directional derivative at the point P.

(a) $D_{\vec{u}} f(P) = \nabla f(P)\cdot\vec{u} = \langle 2x+4\cos(2y+4x),\ 2\cos(2y+4x)\rangle\Big|_P \cdot \left\langle \frac{2}{\sqrt{29}}, \frac{-5}{\sqrt{29}}\right\rangle$

$= \langle \underbrace{4\cos(\pi)}_{-4}, \underbrace{2\cos(\pi)}_{-2}\rangle\cdot\left\langle \frac{2}{\sqrt{29}}, \frac{-5}{\sqrt{29}}\right\rangle$

$= \frac{-8}{\sqrt{29}} + \frac{10}{\sqrt{29}} = \boxed{\frac{2}{\sqrt{29}}}$

(b) $\|\nabla f\|_P = \sqrt{(-4)^2+(2)^2} = \sqrt{20} = \boxed{2\sqrt{5}}$

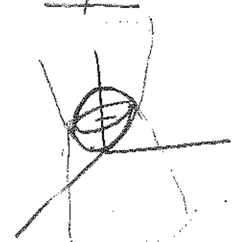
7. (10 pts) Use a triple integral to find the volume between the two paraboloids: $z = x^2 + y^2$ and $z = 2 - x^2 - y^2$.

$$\iint_{\underbrace{x^2+y^2\le 1}_{R}} \int_{x^2+y^2}^{2-x^2-y^2} dz\,dy\,dx$$

Bdry of R: $x^2+y^2 = 2-x^2-y^2$

$2 = 2x^2+2y^2$

$1 = x^2+y^2$



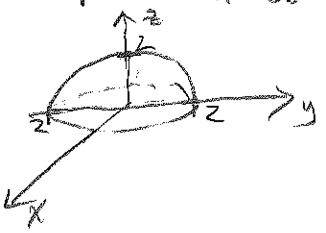
$$= \int_0^{2\pi}\int_0^1\int_{r^2}^{2-r^2} dz\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 z\Big|_{r^2}^{2-r^2} r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \underbrace{(2-r^2-r^2)r}_{2r-2r^3}\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 (2r-2r^3)\,dr\,d\theta = \int_0^{2\pi} r^2 - \frac{2}{4}r^4\Big|_0^1 d\theta = \int_0^{2\pi} 1-\frac{1}{2}\,d\theta = \boxed{\pi}$$

10. (10 pts) Evaluate the integral $\iiint_D (x^2+y^2+z^2)\,dV$, where D is the solid hemisphere $x^2+y^2+z^2 \le 4$, $z \ge 0$.

Use spherical coordinates:



$$\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{2} \rho^2 (\rho^2 \sin\phi)\, d\rho\, d\phi\, d\theta \quad \rho^2$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{2} \rho^4 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \frac{\rho^5}{5}\Big|_0^2 \sin\phi\, d\phi\, d\theta$$

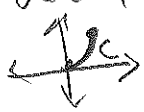
$$= \int_0^{2\pi} \underbrace{-\frac{32}{5}\cos\phi \Big|_{\phi=0}^{\phi=\frac{\pi}{2}}}_{-\frac{32}{5}(\cos(\frac{\pi}{2}) - \cos(0)) = \frac{32}{5}} d\theta$$

$$= \int_0^{2\pi} \frac{32}{5}\, d\theta = \boxed{\frac{64\pi}{5}}$$

11. (10 pts) Compute the line integral $\int_C \vec{F}\cdot d\vec{R}$, where C is the curve $\vec{R}(t) = t\vec{i} + t^2\vec{j}$ and $\vec{F} = y^2\vec{i} + 2xy\vec{j}$ for $0 \le t \le 1$.

C: $\left.\begin{array}{l} x = t \\ y = t^2 \end{array}\right\} y = x^2$ $\quad \vec{F} = \langle y^2, 2xy\rangle = \langle t^4, 2t^3\rangle$

$0 \le t \le 1$ $\quad \vec{R} = \langle t, t^2\rangle$



$d\vec{R} = \langle dt, 2t\,dt\rangle$

$$\vec{F}\cdot d\vec{R} = \langle t^4, 2t^3\rangle \cdot \langle dt, 2t\,dt\rangle$$

$$= t^4\,dt + 4t^4\,dt = 5t^4\,dt$$

$$\int_C \vec{F}\cdot dR = \int_0^1 5t^4\,dt = t^5\Big|_0^1 = \boxed{1}$$

8. (15 pts) Given the function $f(x,y) = x^3 + y^3 + 3x^2 - 18y^2 + 81y + 5$ find all critical points. Then use the second partials test to classify each critical point as a relative maximum, a relative minimum or a saddle point.

Critical points: $f_x = 3x^2 + 6x = 3x(x+2) = 0 \Rightarrow x = 0, x = -2$

$f_y = 3y^2 - 36y + 81 = 3(y^2 - 12y + 27) = 0$

$3(y-3)(y-9) = 0 \Rightarrow y = 3, 9$

$(0,3), (-2,3)$
$(0,9), (-2,9)$

Second Partials Test : $D = f_{xx}f_{yy} - (f_{xy})^2$

$D = (6x+6)(6y-36) - (0)^2$

$(0,3): D = (6)(-18) < 0 \Rightarrow \boxed{(0,3) \text{ Saddle pt}}$

$(-2,3): D = (-6)(-18) > 0,\ f_{xx} < 0\ \cap$ $\boxed{(-2,3) \text{ Rel max}}$

$(0,9): D = (6)(18) > 0,\ f_{xx} > 0\ \cup$ $\boxed{(0,9) \text{ Rel min}}$

$(-2,9): D = (-6)(18) < 0 \Rightarrow \boxed{(-2,9) \text{ Saddle pt}}$

9. (10 pts) Find the surface area of the portion of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 9$. Hint: Use polar coordinates to evaluate the integral.

$$S = \iint_R \sqrt{f_x^2 + f_y^2 + 1}\, dA = \iint_{R:\, x^2+y^2 \le 9} \sqrt{\underbrace{(2x)^2 + (2y)^2 + 1}_{4(x^2+y^2)+1}}\, dA = \int_0^{2\pi}\int_0^3 \sqrt{4r^2+1}\, r\,dr\,d\theta$$

$z = f(x,y) = x^2 + y^2 \Rightarrow f_x = 2x,\ f_y = 2y$

Bdry of R : $x^2 + y^2 = 9$

$$S = \frac{1}{8}\int_0^{2\pi}\int_0^3 \underbrace{(4r^2+1)^{\frac{1}{2}} 8r\,dr}_{\substack{\text{substitution} \\ u^{1/2}du}}\,d\theta = \frac{1}{8}\int_0^{2\pi} (4r^2+1)^{3/2}\frac{2}{3}\Big|_{r=0}^{r=3} d\theta$$

$$= \int_0^{2\pi} \underbrace{\frac{1}{8}\frac{2}{3}\left[(37)^{3/2} - 1\right]}_{\frac{1}{12}(37^{3/2}-1)} d\theta = \frac{2\pi}{12}(37^{3/2} - 1) = \boxed{\frac{\pi}{6}(37^{3/2} - 1)}$$