Lecture 3
Chapter 3 and Review

(1) Homework Problems from Chapter 3 # 2, 3, 5, 6, 7

(2) Additional Notes on Probability of Absorption for RW on $\{0, 1, 2, \ldots, N\}$ and Mean Time Until Absorption

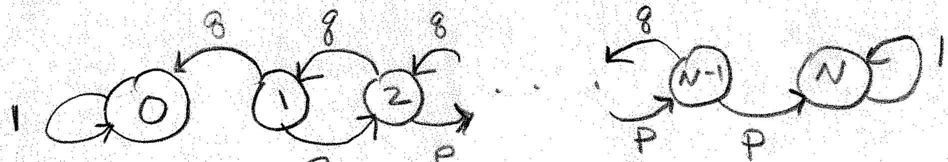
(3) Review Topics for EXAM I.
- Review Lectures and Homework Problems.

Biomathematics II

EXAM #1, Thursday, February 19, Chapters 1, 2, 3 up to 3.5 skip 3.4.3
Homework Problems, February 17, Chapter 3 #1*, 2, 3, 5, 6, 7

Chapter 3 Biological Applications of DTMC

Gambler's Ruin Problem or Random Walk model
on $\{0,1,\ldots,N\}$; $p+q=1$, $p>0, q>0$



$\{0\}$, $\{N\}$, $\{1, 2, \ldots, N-1\}$

positive recurrent aperiodic

transient period = 2

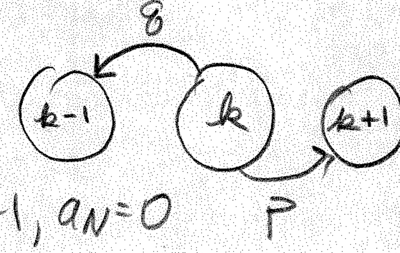
We used the theory of difference equations with constant coefficients to calculate the probability of absorption into states 0 or N

a_k = prob. absorption into state 0 beginning at state k, $k=1,\ldots,N-1$

b_k = prob. absorption into state N beginning at state k, $k=1,\ldots,N-1$

τ_k = expected time until absorption into states 0 or N beginning at state k, $k=1,\ldots,N-1$.

(2)



$$a_k = p a_{k+1} + q a_{k-1}, \quad a_0 = 1, a_N = 0$$

$$b_k = p b_{k+1} + q b_{k-1}, \quad b_0 = 0, b_N = 1$$

$$\tau_k = p(1+\tau_{k+1}) + q(1+\tau_{k-1}), \quad \tau_0 = 0, \tau_N = 1$$

"Backward" equations — Boundary conditions

(1) $p a_{k+1} - a_k + q a_{k-1} = 0, \quad a_0 = 1, \; a_N = 0$

(2) $p b_{k+1} - b_k + q a_{k-1} = 0, \quad b_0 = 0, \; b_N = 1$

(3) $p \tau_{k+1} - \tau_k + q \tau_{k-1} = -1, \quad \tau_0 = 0, \; \tau_N = 1$

These second-order difference equations can be solved by writing the characteristic equation for the homogeneous equation, for (1), (2), (3):

$$p\lambda^2 - \lambda + q = 0 \quad (\text{assume } \lambda^k \text{ is a solution})$$

$$(p\lambda - q)(\lambda - 1) = 0$$

Homog sol: $\left\{ \begin{array}{ll} c_1 (1)^k + c_2 \left(\frac{q}{p}\right)^k & q \neq p \\ c_1 + c_2 k & q = p \end{array} \right\}$

Particular solution to nonhomog. in (3)

p. 115 $\longrightarrow \left\{ \begin{array}{ll} ck & q \neq p \\ ck^2 & q = p \end{array} \right.$ Find C by subsituting into the eq in (3)

③ Summary of Solutions

$$a_k = \begin{cases} \dfrac{\left(\frac{q}{p}\right)^N - \left(\frac{q}{p}\right)^k}{\left(\frac{q}{p}\right)^N - 1}, & p \neq q \\ 1 - \dfrac{k}{N}, & p = q = \frac{1}{2} \end{cases}$$

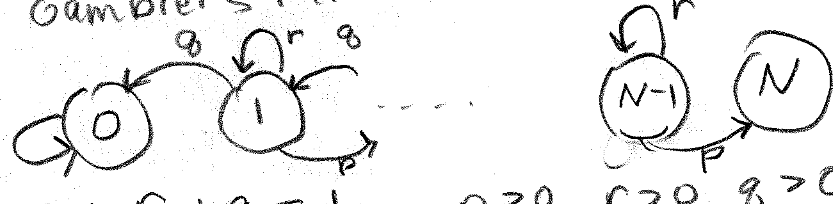
$$b_k = \begin{cases} \dfrac{\left(\frac{q}{p}\right)^k - 1}{\left(\frac{q}{p}\right)^N - 1}, & p \neq q \\ \dfrac{k}{N}, & p = q = \frac{1}{2} \end{cases}$$

$$\boxed{a_k + b_k = 1}$$

$$\tau_k = \begin{cases} \dfrac{1}{q-p}\left[k - N\left(\dfrac{1 - \left(\frac{q}{p}\right)^k}{1 - \left(\frac{q}{p}\right)^N}\right)\right], & p \neq q \\ k(N-k), & p = q = \frac{1}{2} \end{cases}$$

Homework Problems

#6 Repeat this same procedure for Gambler's ruin with "ties"



$p + r + q = 1 \qquad p > 0,\ r > 0,\ q > 0$

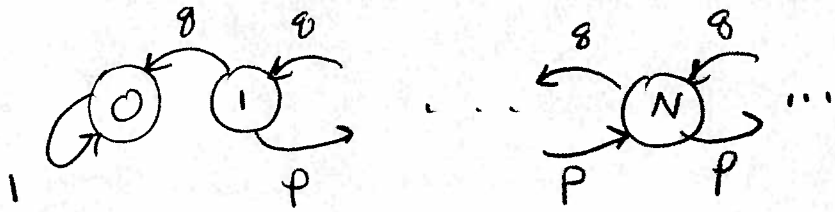
(4) Hint:

$$a_k = p a_{k+1} + r a_k + q a_{k-1}, \quad a_0 = 1, \; a_N = 0$$

$$\tau_k = p(1+\tau_{k+1}) + r(1+\tau_k) + q(1+\tau_{k-1})$$

$$\tau_0 = 0, \; \tau_N = 0$$

#7



Semi-infinite R.W.

$N \to \infty$

(3.4), (3.5), (3.8), (3.9)

Use the formulas in the text for a_k, b_k and τ_k and let $N \to \infty$ to get the formulas for the semi-infinite random walk.
(We talked about this case in class.)

Homework Problems #2, 3, 4, 5 are related to the molecular evolution problem, Example 2.1, p. 49

#2 N virus strains $\{1, 2, \ldots, N\}$ is one comm. class

$$P = \begin{pmatrix} 1-a & \frac{a}{N-1} & \cdots & \frac{a}{N-1} \\ \frac{a}{N-1} & 1-a & \cdots & \frac{a}{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a}{N-1} & \frac{a}{N-1} & \cdots & 1-a \end{pmatrix}$$

a = probability of a mutation to some virus strain

symmetric matrix

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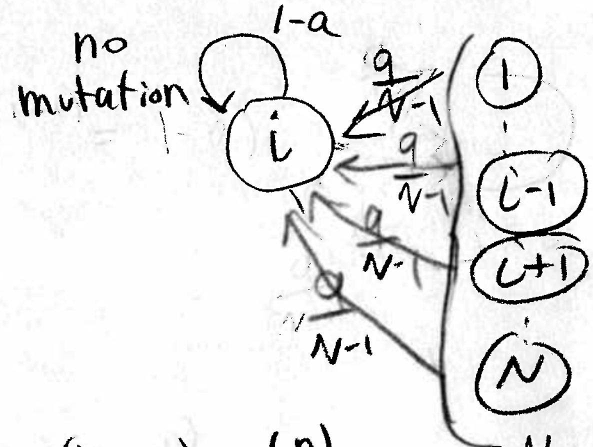
Note: P^2 is symmetric and $p_{ii}^{(2)}$ have the same value for all i and $p_{ij}^{(2)}$ have the same value for all $i \neq j$, i.e.,

$$P^2 = \begin{pmatrix} p_{11}^{(2)} & p_{12}^{(2)} & \cdots & p_{12}^{(2)} \\ p_{12}^{(2)} & p_{11}^{(2)} & \cdots & p_{12}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ p_{12}^{(2)} & p_{12}^{(2)} & \cdots & p_{11}^{(2)} \end{pmatrix}$$

In general $$P^n = \begin{pmatrix} p_{11}^{(n)} & p_{12}^{(n)} & \cdots & p_{12}^{(n)} \\ p_{12}^{(n)} & p_{11}^{(n)} & \cdots & p_{12}^{(n)} \\ \vdots & & \ddots & \vdots \\ p_{12}^{(n)} & p_{12}^{(n)} & \cdots & p_{11}^{(n)} \end{pmatrix}$$

There is only two elements to compute in P^n, namely $p_{11}^{(n)}$ and $p_{12}^{(n)}$.



$$p_{ii}^{(n+1)} = (1-a)\,p_{ii}^{(n)} + \frac{a}{N-1} \sum_{\substack{j=1 \\ j\neq i}}^{N} \underbrace{p_{ij}^{(n)}}_{p_{12}^{(n)}}$$

$$\boxed{p_{11}^{(n+1)} = (1-a)\,p_{11}^{(n)} + a\,p_{12}^{(n)}}$$ — See hints in Appendix

(6)

#3 Special case of #2 with $N=3$

Example 2.1 - Molecular evolution
Jukes-Cantor Model

#4 Use #3 to compute the # generations for a particular mutation sequence to appear

$$\text{Ancestor} \xrightarrow{n\alpha} \text{Descendant}$$

$\alpha = 3a =$ mutation rate = probability of a base (ATGC) substitution per site per generation

$q =$ probability a base substitution has occurred in n generations

$$q = 1 - p_{ii}^{(n)}$$

10 G T A A A T G G C C Ancestor

10 G T $\boxed{T\ T}$ A T G G C C Descendent

$$\frac{2}{10} = 0.2 = q$$

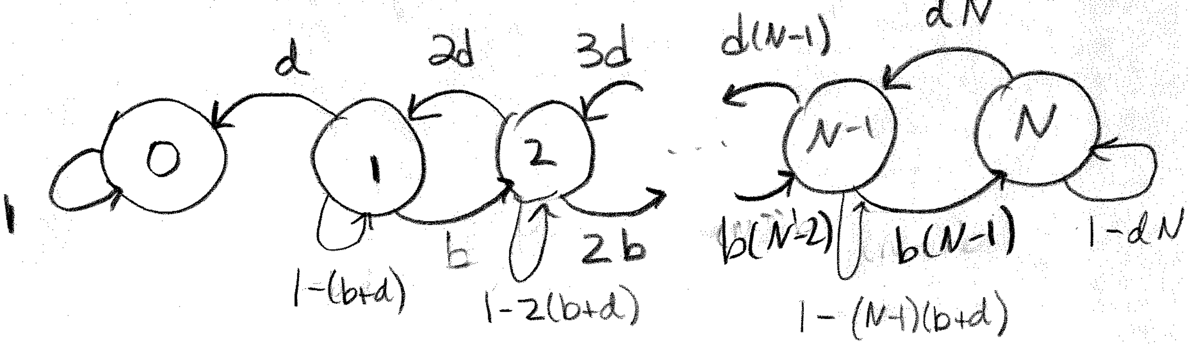
#5 Another molecular evolution model

$$P = \begin{array}{c} \begin{array}{cccc} A & G & C & T \end{array} \\ \begin{pmatrix} p_{11} & a & b & b \\ a & p_{22} & b & b \\ b & b & p_{33} & a \\ b & b & a & p_{44} \end{pmatrix} \end{array} \qquad \begin{array}{l} 0 < a < \frac{1}{3} \\ 0 \le b < \frac{1}{3} \end{array}$$

Base substitution rates are not equally likely.

#16 Simple Birth and Death Process

This is a generalization of the Gambler's Ruin Problem, where the transition probabilities between states depend on the particular state!



$\{0\}$ - absorbing state

$\{1, 2, \ldots, N\}$ - transient states

The method from RW model with constant coeff do NOT work transitions are not the same!!

(8)

$$
P = \begin{array}{c}
\begin{array}{cccccc} 0 & 1 & 2 & & & N \end{array} \\
\begin{pmatrix}
1 & d & 0 & \cdots & & 0 \\
0 & 1-(b+d) & 2d & & & 0 \\
0 & b & 1-2(b+d) & \ddots & & 0 \\
0 & 0 & 2b & \ddots & \ddots & 0 \\
\vdots & & & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & & dN \\
0 & 0 & 0 & \cdots & & 1-dN
\end{pmatrix}
\end{array}
$$

For example,

$$
P = \begin{array}{c}
\begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{pmatrix}
1 & d & 0 & 0 \\
0 & 1-(b+d) & 2d & 0 \\
0 & b & 1-2(b+d) & 3d \\
0 & 0 & 2b & 1-3d
\end{pmatrix}
\end{array}
$$

Assume $(N-1)(b+d) \le 1$ and $Nd \le 1 \Rightarrow$ to ensure P is a stochastic matrix. This assumption puts a restriction on b and d. The <u>generation time</u> $n \to n+1$ is <u>short</u> so that the probability of a birth or a death is <u>small</u>.

(9)

#16 The simple birth and death process from #12 is considered. Let $p_i(n) = \text{Prob}\{X(n)=i\}$. $\{X(n)\}_{n=0}^{\infty}$ is the stochastic process for the size $X(n)$ of the population at generation.

$$p(n) = \begin{pmatrix} p_0(n) \\ p_1(n) \\ \vdots \\ p_N(n) \end{pmatrix}.$$ We use the transition matrix P to compute $p(n+1)$: $Pp(n) = p(n+1)$

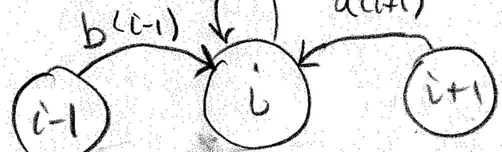
We more "forward" in time $n \to n+1$

$$p_i(n+1) = p_{i-1}(n)\cdot b(i-1) + p_i(n)\left(1 - i(b+d)\right) + p_{i+1}(n)\cdot d(i+1), \quad i = 1, \dots, N-1$$

This is distinct from the first step analysis, where we are considering the end! Note when $i=0$, $p_0(n) = 1$,

$i=1$, $p_1(n+1) = p_1(n)(1-b-d) + p_2(n)\, 2d$

$i=N$, $p_N(n+1) = p_{N-1}(n)\, b(N-1) + p_N(n)\, dN$



The mean popolation size at generation n is $\mu(n)$. How do you define it?

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$$\mu(n) = \sum_{i=0}^{N} i p_i(n)$$

$$p_i(n+1) = p_{i-1}(n) b(i-1) + p_i(n)(1 - i(b+d)) + p_{i+1}(n) d(i+1)$$

$$i p_i(n+1) = p_{i-1}(n) b i(i-1) + p_i(n) i (1 - i(b+d)) + p_{i+1}(n) d i(i+1)$$

$$\underbrace{\sum_{i=0}^{N} i p_i(n+1)}_{\mu(n+1)} = \sum_{i=2}^{N} p_{i-1}(n) b(i-1) i + \sum_{i=1}^{N-1} p_i(n)\, i\,(1 - i(b+d)) + p_N(n) N(1 - dN) + \sum_{i=1}^{N-1} p_{i+1}(n) d i(i+1)$$

$+ \; p$

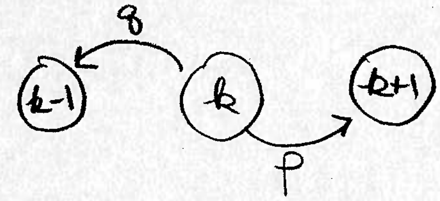
Show

$$\mu(n+1) = (1 + b - d)\mu(n) - bN p_N(n)$$

We will add # 12, #16 to the next homework assignment after the exam!!!

Additional Notes

(11) First-Step analysis:



$$(*) \quad a_k = p a_{k+1} + q a_{k-1}, \quad a_0 = 1, \; a_N = 0$$

$$(**) \quad b_k = p b_{k+1} + q b_{k-1}, \quad b_0 = 0, \; b_N = 1$$

We solved the boundary value problem for a_k. The boundary value problem for b_k is similar. But it is unnecessary to solve for b_k since $a_k + b_k = 1$ - probability of absorption, hitting 0 or N, is one (beginning from any state k)!

Equations (*) and (**) are called "backward equations" rather than "forward equations" because equations (*) and (**) are derived from what happens at the end of the games.

We can write equations (*) and (**) as matrix equations:

$$(*) \quad q a_{k-1} - a_k + p a_{k+1} = 0$$

$$\vec{a} D = \vec{c}, \quad \vec{a} = (a_0, a_1, \ldots, a_N), \quad \vec{c} = (1, 0, \ldots, 0)$$

12

Matrix D is given on p.113, equation (3.6)

$$D = \left(\begin{array}{c|ccccc|c} 1 & q & 0 & 0 & \cdots & 0 & 0 \\ \hline 0 & -1 & q & 0 & \cdots & 0 & 0 \\ 0 & p & -1 & q & \cdots & 0 & \vdots \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 0 \\ \hline 0 & 0 & 0 & 0 & & p & 1 \end{array}\right) = \left(\begin{array}{c|c|c} 1 & A & 0 \\ \hline \vdots & & \\ 0 & T-I & 0 \\ \hline 0 & B & 1 \end{array}\right)$$

$D\vec{a}=\vec{c}$, matrix D is invertible! Why?

(See Definition 2.20 p.75) $\Rightarrow \det(D) = \det(T-I)$

$T-I$ is irreduciby diagonally dominant.

(See Definition 2.20 p.75) $T-I$ is irreducible

and $M=T-I$

$$|m_{ii}| \geq \sum_{\substack{j=1 \\ j\neq i}}^{N-1} |m_{ij}| \quad , \quad i=1,\ldots,N-1$$

with strict inequality for one i.

Matrix $T-I$ is referred to as

<u>irreducibly</u> <u>diagonally</u> <u>dominant</u>.

<u>Examples</u>! $M = \begin{pmatrix} -1 & \frac{1}{2} & -\frac{1}{3} \\ 0 & -1 & 1 \\ 0 & 2 & -2 \end{pmatrix}$ is irreducibly diagonally dominant

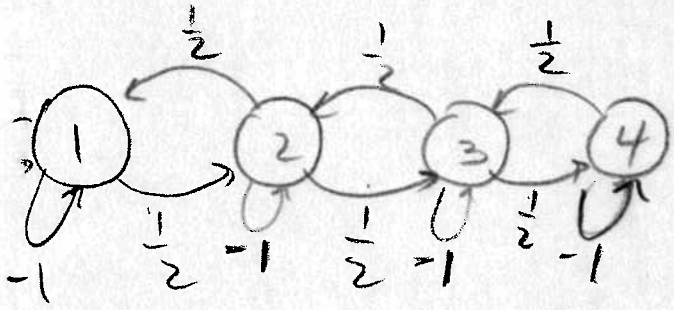
Row 1: $|-1| > |\frac{1}{2}| + |-\frac{1}{3}|$

Row 2: $|-1| = |1|$

Row 3: $|-2| = |2|$

strict inequality for row 1 !!

Example:
$$T - I = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{pmatrix} -1 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -1 \end{pmatrix} \end{array}$$



$T-I$ is irreducible

Row 1: $|-1| > |\frac{1}{2}|$

Row 2: $|-1| = |\frac{1}{2}| + |\frac{1}{2}|$ strict inequality for 2 rows

Row 3: $|-1| = |\frac{1}{2}| + |\frac{1}{2}|$

Row 4: $|-1| > |\frac{1}{2}|$

Therefore, $T-I$ irreducibly diagonally dominant means $T-I$ is invertible (Ortega, 1987) means $\det(T-I) \neq 0$ means D is invertible

$$D^{-1} = \left(\begin{array}{c|c|c} 1 & -A(T-I)^{-1} & 0 \\ \hline 0 & (T-I)^{-1} & 0 \\ \hline 0 & -B(T-I)^{-1} & 1 \end{array}\right)$$

Recall the fundamental matrix $F = (I-T)^{-1}$

So $-A(T-I)^{-1} = AF$ and $-B(T-I)^{-1} = BF$.

(14)

The probability of absorption into state 0 (ruin) is the solution of $\vec{a}D = \vec{c}$ or $\vec{a} = \vec{c}\,D^{-1}$

$$(a_0, a_1, \ldots, a_{N-1}, a_N) = (1, 0, \ldots, 0, 0)\begin{pmatrix} 1 & AF & 0 \\ 0 & (T-I)^{-1} & 0 \\ 0 & BF & 1 \end{pmatrix}$$

$a_0 = 1$, $a_N = 0$

$$\boxed{(a_1, \ldots, a_{N-1}) = AF}$$

The probability of absorption into state N (win) is the solution of $\vec{b}D = \vec{c}$, where $\vec{c} = (0, 0, \ldots, 0, 1)$

$$(b_0, b_1, \ldots, b_{N-1}, b_N) = (0, 0, \ldots, 0, 1)\begin{pmatrix} 1 & AF & 0 \\ 0 & (T-I)^{-1} & 0 \\ 0 & BF & 1 \end{pmatrix}$$

$b_0 = 0$

$b_N = 1$

$$\boxed{(b_1, \ldots, b_{N-1}) = BF}$$

Alternately the "forward equations" can be used to find the probability of absorption

$$P = \begin{array}{c} \begin{matrix} 0 & 1 & 2 & \cdots & & N \end{matrix} \\ \begin{pmatrix} 1 & q & 0 & 0 & \cdots & 0 \\ 0 & 0 & q & 0 & & 0 \\ 0 & p & 0 & q & & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & p & 1 \end{pmatrix} \end{array}$$

Reorder states to put 0 and N first

$$\tilde{P} = \begin{array}{c} \begin{matrix} 0 \; N & \quad 1 \; 2 \cdots N-1 \end{matrix} \\ \left(\begin{array}{c|c} I & \begin{matrix} A \\ B \end{matrix} \\ \hline 0 & T \end{array}\right) \end{array}$$

$$\begin{array}{l} \qquad\;\; 1 \quad 2 \cdots N-1 \\ A = (q, 0, \ldots 0) \\ B = (0, 0, \ldots, p) \end{array}$$

$$\lim_{n \to \infty} \tilde{P}^n = \left(\begin{array}{c|c} I & \begin{matrix} A(I-T)^{-1} \\ B(I-T)^{-1} \end{matrix} \\ \hline 0 & 0 \end{array}\right) = \left(\begin{array}{c|c} I & \begin{matrix} AF \\ BF \end{matrix} \\ \hline 0 & 0 \end{array}\right)$$

probability of absorption into state 0 from $1, 2, \ldots, N-1$ is AF.

probability of absorption into state N from $1, 2, \ldots, N-1$ is BF.

Review for Exam #1

Chapter 1: Discrete vs Continuous random variables X - random variable

Examples of Discrete prob distributions:
- Uniform
- Geometric
- Binomial
- Negative Binomial
- Poisson

Examples of Continuous prob. distributions:
- Uniform
- Gamma
- Exponential
- Normal

Expectation $E(X) = \mu_X$ = mean

$E(u(X))$

$E((X-\mu_X)^2) = \sigma_X^2$ = variance

Probability Generating Function

$$\mathcal{P}_X(t) = E(t^X) = \sum_{j=0}^{\infty} p_j t^j$$

X - discrete
state space $\{0,1,2,..\}$

Moment Generating Function

$$M_X(t) = E(e^{tX}) = \sum_{j=0}^{\infty} p_j e^{tj}$$

X discrete
state space $\{0,1,2,..\}$

$$\text{Prob}\{X=j\} = p_j, \quad j = 0,1,2,\ldots$$

$$\sum_{j=0}^{\infty} p_j = 1$$

Discrete Distributions

<u>Discrete Uniform:</u> $f(x) = \begin{cases} \frac{1}{n}, & x = 1, 2, \ldots, n, \\ 0, & \text{otherwise.} \end{cases}$

$\mu = \frac{1+n}{2}$, $\sigma^2 = \frac{n^2-1}{12}$, $M(t) = \frac{e^{(n+1)t} - e^t}{n(e^t - 1)}$, $t \neq 0$.

<u>Geometric:</u> $f(x) = \begin{cases} p(1-p)^x, & x = 0, 1, 2, \ldots, \\ 0, & \text{otherwise,} \end{cases}$ where $0 < p < 1$.

$\mu = \frac{1-p}{p}$, $\sigma^2 = \frac{1-p}{p^2}$, $M(t) = \frac{p}{1-(1-p)e^t}$.

<u>Binomial $b(n,p)$:</u> $f(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x = 0, 1, 2, \ldots, n, \\ 0, & \text{otherwise,} \end{cases}$ where n is a positive integer and $0 < p < 1$. The notation $\binom{n}{x} = \frac{n!}{x!(n-x)!}$. $\mu = np$, $\sigma^2 = np(1-p)$, $M(t) = (1-p+pe^t)^n$.

<u>Negative Binomial:</u> $f(x) = \begin{cases} \binom{x+n-1}{n-1} p^n (1-p)^x, & x = 0, 1, 2, \ldots, \\ 0, & \text{otherwise,} \end{cases}$ where n is a positive integer and $0 < p < 1$. $\mu = \frac{n(1-p)}{p}$, $\sigma^2 = \frac{n(1-p)}{p^2}$, $M(t) = \frac{p^n}{[1-(1-p)e^t]^n}$.

<u>Poisson, $Po(\lambda)$:</u> $f(x) = \begin{cases} \frac{\lambda^x e^{-\lambda}}{x!}, & x = 0, 1, 2, \ldots, \\ 0, & \text{otherwise,} \end{cases}$ where λ is a positive constant.

$\mu = \lambda$, $\sigma^2 = \lambda$, $M(t) = e^{\lambda(e^t - 1)}$.

Continuous Distributions

<u>Uniform $U(a,b)$:</u> $f(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b, \\ 0, & \text{otherwise,} \end{cases}$ where $a < b$ are constants.

$\mu = \frac{a+b}{2}$, $\sigma^2 = \frac{(b-a)^2}{12}$, $M(t) = \frac{e^{bt} - e^{at}}{t(b-a)}$, $t \neq 0$.

<u>Gamma:</u> $f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, & x \ge 0, \\ 0, & x < 0, \end{cases}$ where α and β are positive and $\Gamma(\alpha) = \int_0^\infty \frac{1}{\beta^\alpha} x^{\alpha-1} e^{-x/\beta}\, dx$.

For a positive integer n, $\Gamma(n) = (n-1)!$. $\mu = \alpha\beta$, $\sigma^2 = \alpha\beta^2$, $M(t) = \frac{1}{(1-\beta t)^\alpha}$, $t < 1/\beta$.

<u>Exponential:</u> $f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \ge 0, \\ 0, & x < 0, \end{cases}$ where λ is a positive constant.

$\mu = \frac{1}{\lambda}$, $\sigma^2 = \frac{1}{\lambda^2}$, $M(t) = \frac{\lambda}{\lambda - t}$, $t < \lambda$.

<u>Normal, $N(\mu, \sigma^2)$:</u> $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$, $-\infty < x < \infty$, where μ and σ are constants.

$E(X) = \mu$, $Var(X) = \sigma^2$, $M(t) = e^{\mu t + \sigma^2 t^2/2}$.

The probability generating function (p.g.f.) is $\mathcal{P}_X(t) = E(t^X)$, moment generating function (m.g.f.) is $M_X(t) = E(e^{tX})$ and cumulant generating function (c.g.f.) is $K_X(t) = \ln(M_X(t))$.

Cumulant Generating Function

$$K_X(t) = \ln(M_X(t))$$

Formulas, bottom p. 21

$$\mu_X = P_X'(1) = M_X'(0) = K_X'(0)$$

$$\sigma_X^2 = \begin{cases} P_X''(1) - P_X'(1) - [P_X'(1)]^2 \\ M_X''(0) - [M_X'(0)]^2 \\ K_X''(0) \end{cases}$$

What is a Stochastic Process?

Collection of random variables defined on two sets - Sample space and time

$$\{X(t,s) \mid t \in T, s \in S\}$$

Often the sample space S is omitted

$\{X(t)\}_{t\in T}$ $\quad T = [0,\infty)$ cont. $\quad T = \{0,1,2,\dots\}$ discrete

$X(t)$ cont $\quad X(t)$ discrete

Chapters 2, 3, 4 $\quad \{X(n)\}_{n=0}^{\infty}$ $\quad X(n)$ - discrete, $T = \{0,1,\dots\}$ discrete

Chapters 5, 6, 7 $\quad \{X(t)\}_{t\in[0,\infty)}$ $\quad X(t)$ - discrete, $T = [0,\infty)$ cont.

Chapters 8, 9 $\quad \{X(t)\}_{t\in[0,\infty)}$ $\quad X(t)$ - cont, $T = [0,\infty)$ - cont

Chapter 2 DTMC

$\{X(n)\}_{n=0}^{\infty}$ State Space $\{0,1,2,..,N\}$

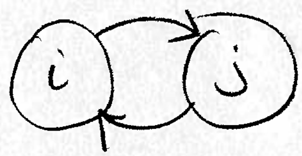
$\{0,\pm 1,\pm 2,...\}$

$\{0,1,2,...\}$

etc

Transition Matrix P is a stochastic matrix

Digraph:



$i \leftrightarrow j$

Communicating class: $\exists$ path $i \to j$ and $j \to i$ $\forall\, i,j$ in the class

Reducible vs Irreducible chain

Periodic vs Aperiodic Class

Recurrent vs Transient class

Positive or Null Recurrent

A finite DTMC can either have positive recurrent or transient classes. Only infinite DTMC can have null recurrent or positive recurrent or transient classes

ergodic $\Rightarrow$ $\exists$ stationary distribution $\Rightarrow$ $\boxed{P\pi = \pi}$

Basic Limit Theorems for Recurrent classes:

$$\begin{cases} \text{Aperiodic} & \lim_{n\to\infty} p_{ij}^{(n)} = \dfrac{1}{\mu_{ii}} \\ \text{Periodic (period } d\text{)} & \lim_{n\to\infty} p_{ii}^{(nd)} = \dfrac{d}{\mu_{ii}} \end{cases}$$

$\mu_{ii} = \sum_{i=1}^{\infty} i f_{ii}^{(n)}$

mean recurrence time

(19)

Chapter 3 Applications

Proliferating Epithelial Cells
Gambler's Ruin or RW model on $\{0,1,\ldots,N\}$
RW Model on $\{0,1,2,\ldots\}$

Probability of Absorption: a_k, b_k
First Step Analysis
Mean Time Until Absorption: τ_k

Skip Section 3.4.3

a_{kn} = probability of absorption into state 0 from state k at nth step.
b_{kn} = probability of absorption into state N from state k at nth step

$$A_k(t) = \sum_{n=0}^{\infty} a_{kn} t^n, \quad A_0(t)=1 \quad A_N(t)=0$$

$$B_k(t) = \sum_{n=0}^{\infty} b_{kn} t^n, \quad B_0(t)=0 \quad B_N(t)=1$$

$$A_k(1) = \sum_{n=0}^{\infty} a_{kn} = a_k$$

$$B_k(1) = \sum_{n=0}^{\infty} b_{kn} = b_k$$

$A_k(t)$, $B_k(t)$ - closed form expressions by solving difference equations!