## Chapter 4: Discrete-Time Branching Processes

Generating functions rather than transition matrices are useful in analysis of branching processes. Offspring p.g.f.:

$$f(t) = \sum_{k=0}^{\infty} p_k t^k$$

Recall $f(1) = 1$ and $m = f'(1)$ is the mean number of offspring. The branching process is called *subcritical* if $m < 1$, *critical* if $m = 1$, and *supercritical* if $m > 1$.

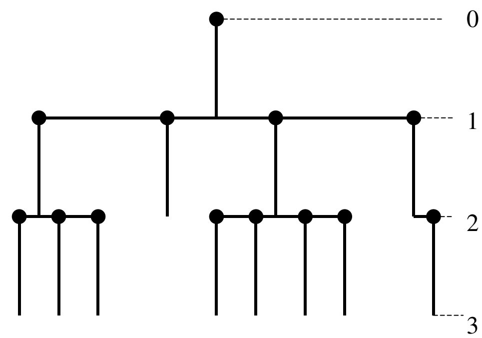


Figure 1: Sample path of a branching process $\{X_n\}_{n=0}^{\infty}$. In the first generation, four individuals are born, $X_1 = 4$. The four individuals in generation one give birth to three, zero, four, and one individuals, respectively, making a total of eight individuals in generation two, $X_2 = 8$.

**Theorem 1 (Branching Process Theorem)** *Let $X_0 = 1$. Assume $f(0) = p_0 > 0$ and $p_0 + p_1 < 1$.*

*(i) If $m \le 1$, then $\lim_{n\to\infty} \text{Prob}\{X_n = 0\} = 1$.*

*(ii) If $m > 1$, then $\lim_{n\to\infty} \text{Prob}\{X_n = 0\} = q$, where $q = f(q)$ is the unique fixed point in the interval $(0, 1)$.*

To verify this theorem, we show that given $X_0 = 1$, then $\mathcal{P}_{X_n}(t) = f^{n-1}(f(t)) = f^n(t)$, the p.g.f. of $X_n$ is the n-fold composition of the offspring p.g.f. Then $p_0(n) = \mathcal{P}_{X_n}(0) = f^n(0)$ and $p_0(n) = f(p_0(n-1))$. The sequence $\{p_0(n)\}$ is an increasing function, bounded above by one and therefore, has a limit $q$. This limit is a fixed point of $f$, $f(q) = q$.

If $X_0 = N$ and $m > 1$, then $\lim_{n\to\infty} p_0(n) = \lim_{n\to\infty} \text{Prob}\{X_n = 0\} = q^N$. The conditional expectation $E(X_{n+1}|X_n) = mX_n$, $E(X_{n+1}) = mE(X_n)$. If the growth rate varies with $n$, with a mean $m_n$, then $E(X_{n+1}) = m_n E(X_n)$.

**Multitype Branching Process, $k$ different types:** $X(n) = (X_1(n) \ldots, X_k(n))$. The offspring random variable for parent of type $i$ with offspring of type $l$ is $Y_{il}$. Offspring p.g.f.s:

$$f_i(t_1, \ldots, t_k) = \sum_{s_k}^{\infty} \cdots \sum_{s_1}^{\infty} P_i(s_1, \ldots, s_k) t_1^{s_1} \cdots t_k^{s_k}$$

$$P_i(s_1, \ldots, s_k) = \text{Prob}\{Y_{i1} = s_1, \ldots, Y_{ik} = s_k\}.$$

We assume not all of the offspring p.g.f.s are simple, where *simple function* means that $f_i(0, \ldots, 0) = 0$ and $f_i$ is linear in the $t$ variables. Expectation matrix $M = (m_{ij})$, where

$$m_{ji} = \left.\frac{\partial f_i}{\partial t_j}\right|_{t_1=1,\ldots,t_k=1}$$

Assume $M$ is irreducible. Denote the spectral radius of $M$ as $\rho(M)$, the maximum modulus of the eigenvalues of $M$.The multitype branching process is called *subcritical* if $\rho(M) < 1$, *critical* if $\rho(M) = 1$ and *supercritical* if $\rho(M) > 1$. In addition, if matrix $M$ is regular, some positive integer power of $M$ is strictly positive, $M^{n_0} > 0$, then the following theorem holds.

**Theorem 2** *If not all p.g.f.s are simple, matrix $M$ is regular and $X_i(0) = r_i \geq 0$, then the probability of extinction depends on $\rho(M) = \lambda$.*

*(i) If $\rho(M) \leq 1$, then $\lim_{n\to\infty} \text{Prob}\{X(n) = 0\} = 1$.*

*(ii) If $m > 1$, then $\lim_{n\to\infty} \text{Prob}\{X(n) = 0\} = q_1^{r_1} \cdots q_k^{r_k}$ where $q_i = f_i(q_1, \ldots, q_k)$, $i = 1, \ldots, k$ is the unique fixed point in the interval $[0, 1)$.*

In addition, the expectation of the process satisfies $E(X(n+1)) = ME(X(n))$ or $E(X(n)) = M^n E(X(0))$. Thus, even though on the average the population may increase geometrically, $\rho(M) > 1$, there is still a positive probability of extinction (part (ii) of the theorems). See Chapter on Branching Processes.

**Chapter 5: Continuous-Time Markov Chains (CTMCs)**

Discrete random variable $X(t)$, $t \in [0, \infty)$. Probabilities $p_i(t) = \text{Prob}\{X(t) = i\}$. Transition probability: $p_{ji}(t, s) = \text{Prob}\{X(t) = j | X(s) = i\}$, $s < t$. We will assume *time-homogenous* transition probabilities $p_{ji}(t, s) = p_{ji}(t - s)$. The transition matrix is a stochastic matrix:

$$P(t) = (p_{ij}(t)) = \begin{pmatrix} p_{00}(t) & p_{01}(t) & \cdots \\ p_{10}(t) & p_{11}(t) & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix},$$

where

$$p_{ji}(\Delta t) = \delta_{ji} + q_{ji}\Delta t + o(\Delta t)$$

is an infinitesimal transition probability. The *infinitesimal generator matrix*:

$$Q = (q_{ij}) = \begin{pmatrix} q_{00} & q_{01} & \cdots \\ q_{10} & q_{11} & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix}, \quad Q = \lim_{\Delta t \to 0} \frac{P(\Delta t) - I}{\Delta t}$$

The column sums of $Q$ equal zero.

Forward Kolmogorov differential equations: $\dfrac{dP(t)}{dt} = QP(t)$

Backward Kolmogorov differential equations: $\dfrac{dP(t)}{dt} = P(t)Q$

In physics and chemistry, the forward Kolmogorov differential equations are often referred to as the *Master equation.*

The *embedded DTMC* is used to define irreducible, recurrent, and transient states or chains for the associated CTMC. Let $Y_n$ denote the random variable for the state of a CTMC $\{X(t) : t \in [0, \infty)\}$ at the time of the $n$th jump, $Y_n = X(W_n)$. (See Figure 2.) The set of discrete random variables $\{Y_n\}_0^\infty$ is the *embedded Markov chain.*

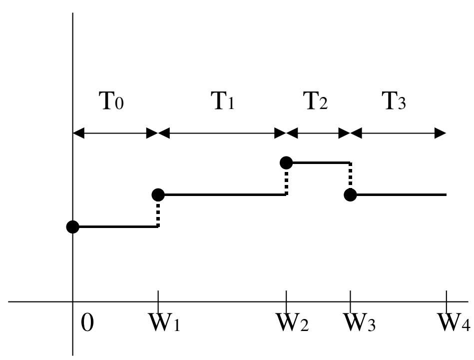


Figure 2: Sample path of a CTMC, illustrating waiting times $\{W_i\}$ and interevent times, $\{T_i\}$.

A CTMC is irreducible, recurrent or transient if the corresponding embedded Markov chain has these properties. Some differences in the dynamics of a CTMC as opposed to a DTMC are the possibility of a finite-time blow up in a CTMC (*explosive process*) and the fact that CTMC are not periodic. See Figure 3. The embedded MC cannot be used to classify chains as positive recurrent or null recurrent. This latter classification depends on the mean recurrence time $\mu_{ii}$.

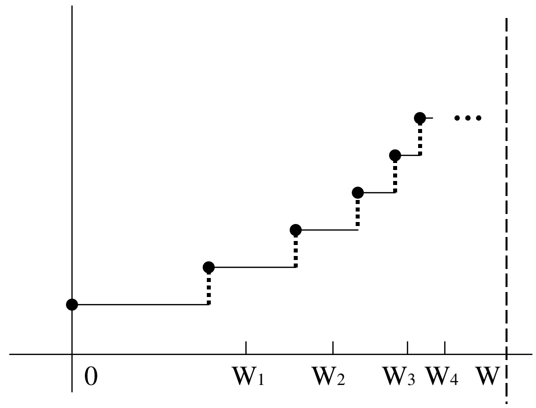


Figure 3: One sample path of a continuous time Markov chain that is explosive.

**Theorem 3 (Basic Limit Theorem for CTMCs)** *If the CTMC $\{X(t) : t \in [0, \infty)\}$ is nonexplosive and irreducible, then for all $i$ and $j$,*

$$\lim_{t\to\infty} p_{ij}(t) = -\frac{1}{q_{ii}\mu_{ii}}, \tag{1}$$

*where $\mu_{ii}$ is the mean recurrence time, $0 < \mu_{ii} \leq \infty$. In particular, a finite, irreducible CTMC is nonexplosive and the limit (1) exists and is positive.*

If the DTMC is nonexplosive and positive recurrent, it has a limiting positive stationary distribution $\pi$ satisfying $Q\pi = \mathbf{0}$, where

$$\pi_i = -\frac{1}{q_{ii}\mu_{ii}}.$$

**Poisson process** with $X(0) = 0$, $p_{i+1,i}(\Delta t) = \lambda\Delta t + o(\Delta t)$ and $p_i(t) = e^{-\lambda t}(\lambda t)^i/i!$ has generator matrix

$$Q = \begin{pmatrix} -\lambda & 0 & 0 & \cdots \\ \lambda & -\lambda & 0 & \cdots \\ 0 & \lambda & -\lambda & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

The associated embedded Markov chain has a transition matrix

$$T = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

The Poisson process is transient.

**A finite CTMC with two states** $\{1, 2\}$. The generator matrix

$$Q = \begin{pmatrix} -a & b \\ a & -b \end{pmatrix}, \quad a, b > 0$$

The CTMC is irreducible and positive recurrent. The limiting stationary distribution can be found from the forward Kolmogorov differential equations by solving for the stationary distribution $Q\pi = \mathbf{0}$. In this case $\pi = (b/(a+b), a/(a+b))^{tr}$. The mean recurrence times are $\mu_{ii} = \dfrac{a+b}{ab}$, $i = 1, 2$.

To generate sample paths, we must know the time between jumps and the state to which the process jumps. The Markov assumption implies the interevent time is exponentially distributed because the exponential distribution has the "memoryless property". Let $T_i$ be the continuous random variable for the time until the $i+1$st event. See Figure 2.

**Theorem 4 (Interevent Time)** *Assume* $\sum_{j\neq n} p_{jn}(\Delta t) = \alpha(n)\Delta t + o(\Delta t)$*. Then the cumulative distribution function for the interevent time* $T_i$ *is* $F_i(t) = 1 - \exp(-\alpha(n)t)$ *with mean and variance*

$$\mu_{T_i} = \frac{1}{\alpha(n)} \quad \text{and} \quad \sigma^2_{T_i} = \frac{1}{[\alpha(n)]^2}$$

**Theorem 5 (Interevent Time Simulation)** *Let* $U \sim U[0,1]$ *be the uniform distribution on [0,1] and* $T_i$ *the continuous random variable for interevent time with state space* $[0,\infty)$*. Then*

$$T_i = F_i^{-1}(U) = -\frac{\ln(U)}{\alpha(n)}.$$

**Simple Birth and Death Markov Chain:** In the simple birth and death process, an event can be a birth or a death. Let $X(0) = N$. The infinitesimal transition probabilities are

$$\begin{aligned} p_{i+j,i}(\Delta t) &= \text{Prob}\{\Delta X(t) = j | X(t) = i\} \\ &= \begin{cases} \mu i\,\Delta t + o(\Delta t), & j = -1 \\ \lambda i\,\Delta t + o(\Delta t), & j = 1 \\ 1 - (\lambda+\mu) i\,\Delta t + o(\Delta t), & j = 0 \\ o(\Delta t), & j \neq -1,\ 0,\ 1. \end{cases} \end{aligned}$$

Use two random numbers, $u_1$ and $u_2$, from the uniform distribution $U(0,1)$ to determine the interevent time and the state to which the process jumps. In MATLAB, indices begin from 1, so instead of writing $t(0)$, we use $t(1)$. Consider the simple birth and death chain, in a MATLAB program, $t(1) = 0$, and the time to the next event is $t(2) = t(1) + \ln(u_1)/(\alpha(n))$, where $\alpha(n) = \lambda n + \mu n$, given the process is in state $n$. Since there are two events, to determine whether there is a birth or a death, the unit interval is divided into two subintervals, one subinterval has probability $\lambda/(\lambda+\mu)$ and the other has probability $\mu/(\lambda+\mu)$. Generate a uniform random number $u_2$. If $u_2 < \lambda/(\lambda+\mu)$, then this random number lies in the first subinterval and there is a birth, otherwise if $u_2 > \lambda/(\lambda+\mu)$, the random number lies in the second subinterval and there is a death. This concept can be easily extended to $k > 2$ events. In a MATLAB program with $k$ events the unit interval must be divided into $k$ subintervals, each with predetermined probability for $i = 1, \ldots, k$ that depends on the current state and the transition probabilities. For example, suppose there are four events with the following rates $a_i(n)$, $i = 1, 2, 3, 4$ which depend on the current state $n$. The probabilities of these four events are $a_i(n)/a(n)$, $a(n) = \sum_i a_i(n)$, $i = 1, 2, 3, 4$. The subinterval $[0, 1]$ is subdivided into four subintervals with the following endpoints:

$$0,\ \frac{a_1}{a},\ \frac{a_1+a_2}{a},\ \frac{a_1+a_2+a_3}{a},\ 1.$$

Therefore, in a MATLAB program:

```
if u2<a1/a, then event 1 occurs
elseif u2>=a1/a & u2<(a1+a2)/a, then event 2 occurs
elseif u2>=(a1+a2)/2 & u2<(a1+a2+a3)/a, then event 3 occurs
else u2>=(a1+a2+a3)/a, then event 4 occurs.
```

The subintervals change each time the process changes state $n$. If the number of events are large, deciding which event occurs can become quite lengthy and there are ways to speed up the process of selecting a particular event.

```
%MatLab program: simple birth and death process
clear all
x0=5; b=1; d=0.5;  % initial and parameter values
for j=1:3   % Three sample paths
clear x t
n=1; t(1,j)=0; x(n)=x0;  % starting values
while x(n)>0 & x(n)<50;   % continue until the process hits zero or reaches 50
    u1=rand; u2=rand;    % two uniform random numbers
    t(n+1,j)=-log(u1)/(b*x(n)+d*x(n))+t(n,j);
    if u2< b/(b+d);
         x(n+1)=x(n)+1;
    else
         x(n+1)=x(n)-1;
    end
    n=n+1;
end
s=stairs(t(:,j),x,'r-','Linewidth',2);
hold on
end
xlabel('Time'); ylabel('Population size');
hold off
```

Figure 4: Three sample paths of the simple birth and death process, $X(0) = 5$, $\lambda = b = 1$, $\mu = d = 0.5$.