Objectives:

1. Use the definition of the Itô Stochastic integral to verify the value of a stochastic integral

$$\int_a^b f(t)\,dW(t) = \underset{k\to\infty}{\text{l.i.m.}} \sum_{i=1}^{k} f(t_i)\,\Delta W(t_i)$$

2. Define Itô's formula which can is the short hand method for computing <u>some</u> Itô stochastic integrals.

3. Relate the forward Kolmogorov differential equation for a diffusion process to an Itô SDE = Itô diffusion process.

$X(t)$ $\quad dX(t) = \alpha(X(t),t)\,dt + \beta(X(t),t)\,dW(t)$ <u>sample paths</u>

$p(x,t)$ $\quad \dfrac{\partial p}{\partial t} = -\dfrac{\partial(\alpha(x,t)p)}{\partial x} + \dfrac{1}{2}\dfrac{\partial^2(\beta^2(x,t)p)}{\partial x^2}$ <u>p.d.f</u>

$\alpha(x,t)$ = infinitesimal mean
$\beta^2(x,t)$ = infinitesimal variance

4. Euler-Maruyama Method for Numerically solving Itô SDEs

8.7 Itô Stochastic Integral

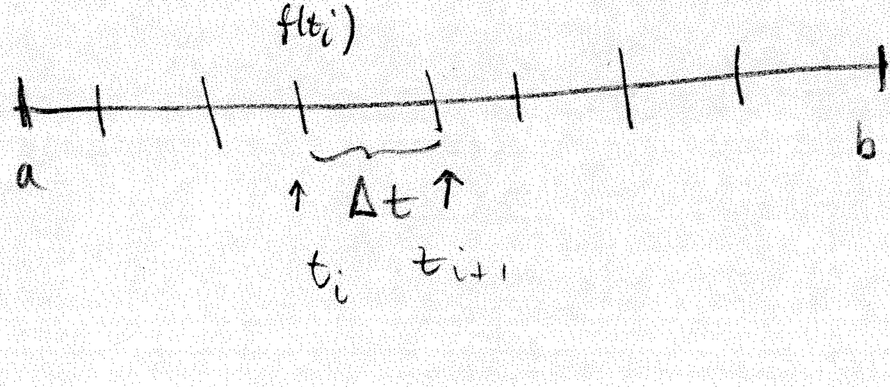
Assume we have a random function $f(X(t),t)$, where $X(t)$ is a continuous random variable. Let $f(t) \equiv f(X(t),t)$ and the following expectation is finite:

$$\int_a^b E(f^2(t))\,dt < \infty$$

Where the preceding interal is a Riemann integral. We defined an Itô Stochastic integral as follows:

$$\int_a^b f(t)\,dW(t) = \underset{k \to \infty}{\text{l.i.m.}} \sum_{i=1}^{k} f(t_i)\,\Delta W(t_i)$$

Where the interval $[a,b]$ is divided into k equal intervals of length $\Delta t = t_{i+1} - t_i = \frac{b-a}{k}$, $\Delta W(t_i) = W(t_{i+1}) - W(t_i)$, and f is evaluated at the left endpoint



Also, the notation $\underset{k\to\infty}{\text{l.i.m.}}$ denotes "mean square convergence

$I = \int_a^b f(t)\,dW(t)$ is the answer and the finite sum is denoted $F_k = \sum_{i=1}^{k} f(t_i)\,\Delta W(t_i)$,

then $\underset{k\to\infty}{\text{l.i.m.}}\, F_k = I$ means

$$\lim_{k\to\infty} E\left[\left(F_k - I\right)^2\right] = 0.$$

We will go through <u>one</u> example using this definition to compute an Itô stochastic integral. Then we will show a shortcut known as <u>Itô's formula</u> to compute the value of an Itô stochastic integral for those cases where it can be computed!

<u>Example: 8.4</u> We will show, using the definition that

$$\boxed{\int_0^t W(t)\,dW(t) = \frac{1}{2}\left[W^2(t) - t\right]}$$

Note: $\int E(W^2(t))\,dt = \int_a^b t\,dt = \frac{b^2-a^2}{2} < \infty$

It is fortunate that we know the answer. $I = \frac{1}{2}\left[W^2(t) - t\right]$, Therefore, we want to show for $F_k = \sum_{i=1}^{k} W(t_i)\,\Delta W(t_i)$ that

$$\lim_{k \to \infty} E\left[(F_k - I)^2\right] = 0$$

or

$$\lim_{k \to \infty} E\left[\left(\sum_{i=1}^{k} W(t_i)\,\Delta W(t_i) - \frac{1}{2}\left[W^2(t) - t\right]\right)^2\right] = 0$$

We will simplify the notation:

$$\Delta W(t_i) = W(t_{i+1}) - W(t_i) = W_{i+1} - W_i$$

$$W(t_i) = W_i$$

$$\Delta t = \frac{t - 0}{k} = \frac{t}{k}$$

First, we obtain a simplified expression for $\sum_{i=1}^{k} W_i\,\Delta W_i$:

$$\begin{aligned}
\Delta(W_i^2) &= W_{i+1}^2 - W_i^2 \\
&= (W_{i+1} - W_i)^2 + 2W_{i+1}W_i - 2W_i^2 \\
&= (\Delta W_i)^2 + 2W_i(W_{i+1} - W_i)
\end{aligned}$$

$$\Delta(W_i^2) = (\Delta W_i)^2 + \underbrace{2W_i\,\Delta W_i}$$

So

$$\boxed{W_i\,\Delta W_i = \frac{1}{2}\Delta(W_i^2) - \frac{1}{2}(\Delta W_i)^2}$$

Sum from $i=1$ to $i=k$:

$$\sum_{i=1}^{k} W_i \Delta W_i = \frac{1}{2}\sum_{i=1}^{k}\left(\Delta(W_i^2) - (\Delta W_i)^2\right)$$

$$= \frac{1}{2}\left[\cancel{W_2^2} - W_1^2 + \cancel{W_3^2} - \cancel{W_2^2} + \cdots + W_{k+1}^2 - \cancel{W_k^2}\right] - \frac{1}{2}\sum_{i=1}^{k}(\Delta W_i)^2$$

$$= \frac{1}{2}W_{k+1}^2 - \underbrace{\overset{0}{\cancel{\frac{1}{2}W_1^2}}} - \frac{1}{2}\sum_{i=1}^{k}(\Delta W_i)^2$$

But $W_1 = W(t_1) = W(0) = 0$ and $W_{k+1}^2 = W^2(t)$

So,

$$\sum_{i=1}^{k} W_i \Delta W_i = \frac{1}{2}W^2(t) - \frac{1}{2}\sum_{i=1}^{k}(\Delta W_i)^2$$

Also,

$$F_k - I = \left[\sum_{i=1}^{k} W_i \Delta W_i\right] - \left[\frac{1}{2}W^2(t) - \frac{t}{2}\right]$$

$$= \left[\underbrace{\frac{1}{2}W^2(t)} - \frac{1}{2}\sum_{i=1}^{k}(\Delta W_i)^2\right] - \left[\underbrace{\frac{1}{2}W^2(t)} - \frac{t}{2}\right]$$

$$= \frac{1}{2}\left[t - \sum_{i=1}^{k}(\Delta W_i)^2\right]$$

Therefore, we just need to show

$$\lim_{k\to\infty} E\left[\left(t - \sum_{i=1}^{k}(\Delta W_i)^2\right)^2\right] = 0$$

Show $E\Big[\underbrace{\Big(t-\sum_{i=1}^{k}(\Delta W_i)^2\Big)^2}\Big] \to 0$ as $k\to\infty$

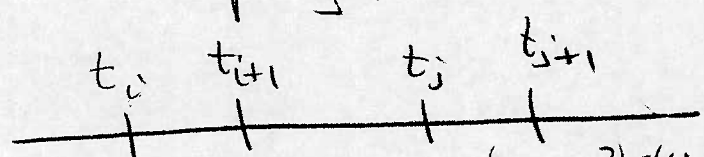
$$E\left(t^2 - 2t\sum_{i=1}^{k}(\Delta W_i)^2 + \sum_{i=1}^{k}(\Delta W_i)^2\sum_{j=1}^{k}(\Delta W_j)^2\right)$$

(*) $$\underbrace{= t^2 - 2t\sum_{i=1}^{k}E(\Delta W_i)^2 + E\left(\sum_{i=1}^{k}(\Delta W_i)^2\sum_{j=1}^{k}(\Delta W_j)^2\right)}$$

linearity of the expectation

$\Delta W_i = W(t_{i+1}) - W(t_i) \sim N(0, t_{i+1}-t_i) = N(0,\Delta t)$

ΔW_i and ΔW_j are independent if $i\neq j$ since they are on nonoverlapping intervals



$$E\left((\Delta W_i)^2(\Delta W_j)^2\right) = \begin{cases} E\left((\Delta W_i)^2\right)E\left((\Delta W_j)^2\right), & i\neq j \\ E\left((\Delta W_i)^4\right), & i=j \end{cases}$$

$$\boxed{\begin{aligned} E\left((\Delta W_i)^4\right) &= 3(\Delta t)^2 \\ E\left((\Delta W_i)^2\right) &= \Delta t \end{aligned}}$$

= 4th moment of the normal distribution $\Delta W_i \sim N(0,\Delta t)$

Thus

$$E\Big[\Big(t^2-\sum_{i=1}^{k}(\Delta W_i)^2\Big)^2\Big] \overset{(*)}{=} t^2 - 2t\sum_{i=1}^{k}\Delta t + \sum_{i=1}^{k}\sum_{\substack{j=1\\ j\neq i}}^{k}\underbrace{E\left((\Delta W_i)^2\right)}_{\Delta t}\underbrace{E\left((\Delta W_j)^2\right)}_{\Delta t}$$

$$\sum_{i=1}^{k}\underbrace{E\left((\Delta W_i)^4\right)}_{3(\Delta t)^2},$$

$$\boxed{\Delta t = \frac{t}{k}}$$

Therefore, substituting $\Delta t = \frac{t}{k}$,

$$E\left[\left(t^2 - \sum_{i=1}^{k} (\Delta W_i)^2\right)^2\right] = t^2 - 2t\left(\sum_{i=1}^{k} \frac{t}{k}\right) + \sum_{i=1}^{k} \sum_{\substack{j=1 \\ j \neq i}}^{k} \left(\frac{t}{k}\right)^2 + \sum_{i=1}^{k} 3\left(\frac{t}{k}\right)^2$$

$$= t^2 - 2t(t) + k(k-1)\frac{t^2}{k^2} + \frac{3t^2}{k^2}k$$

$$= \underbrace{t^2 - 2t^2 + t^2}_{0} \underbrace{- \frac{t^2}{k} + \frac{3t^2}{k}} \to 0 \text{ as } k \to \infty.$$

Thus,

$$\boxed{\int_0^t W(\tau)\,dW(\tau) = \frac{1}{2}\left[W^2(t) - t^2\right]}$$

Example 8.5

$$\int_a^b W(\tau)\,dW(\tau) = \frac{1}{2}\left[W^2(\tau) - \tau\right]\Bigg|_{\tau=a}^{\tau=b}$$

$$= \frac{1}{2}\left[W^2(b) - W^2(a)\right] - \frac{1}{2}(b-a)$$

Example 8.6

$$E\left[\left(\int_a^b W(t)\,dW(t)\right)^2\right] \overset{\text{Itô Isometry}}{=} \int_a^b E(W^2(t))\,dt$$

$$= \int_a^b t\,dt = \frac{1}{2}(b^2 - a^2)$$

8.8 Itô SDE

We have been working with integrals, but generally the stochastic processes of Itô type are written as stochastic differential equations.

$$X(t) = X(0) + \underbrace{\int_0^t \alpha(X(\tau),\tau)\,d\tau}_{\text{Riemann}} + \underbrace{\int_0^t \beta(X(\tau),\tau)\,dW(\tau)}_{\text{Itô stochastic}}$$

is an <u>Itô stochastic integral equation.</u>

$$\boxed{dX(t) = \alpha(X(t),t)\,dt + \beta(X(t),t)\,dW(t)}$$

is an Itô stochastic differential equation which we abbreviate as <u>Itô SDE</u>.

Note the SDE is only for notational convenience, since $W(t)$ is continuous but nowhere differentiable!

Some mathematical assumptions are required for <u>existence and uniqueness</u> of sample paths – see p. 381

(a) $|\alpha(x,t) - \alpha(y,t)| + |\beta(x,t) - \beta(y,t)| \le K|x-y|$
for $x,y \in \mathbb{R}$, $t \in [0,T]$

(b) $|\alpha(x,t)|^2 + |\beta(x,t)|^2 \le K^2(1+|x|^2)$,
for $x \in \mathbb{R}$, $t \in [0,T]$

2 Itô's Formula

$$\boxed{dX(t) = \alpha(X(t),t)dt + \beta(X(t),t)dW(t)}$$

Let $F(x,t)$ be real-valued and defined for $x \in \mathbb{R}$, $t \in [a,b]$ with continuous partial derivatives, $\frac{\partial F}{\partial t}$, $\frac{\partial F}{\partial x}$, $\frac{\partial^2 F}{\partial x^2}$, then

$$\boxed{dF(X(t),t) = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial x}dX + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\underbrace{(dX)^2}}$$

$$(dX)^2 = \alpha^2 \cancel{(dt)^2}^{\,0} + \alpha\beta \cancel{dt\,dW}^{\,0} + \underbrace{\beta^2 (dW)^2}_{\beta^2 dt}$$

$$\boxed{(dX)^2 = \beta^2 dt}$$

Example 8.9 Evaluate $\int_a^b W(t)\,dW(t)$

$$X(t) = \int_a^t W(\tau)\,dW(\tau) \Rightarrow dX(t) = \underbrace{W(t)}\,dW(t)$$

Let $F(w,t) = w^2$

$$dF(W,t) = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial w}dW + \frac{1}{2}\frac{\partial^2 F}{\partial w^2}\underbrace{(dW)^2}_{dt}$$

$$= 0\,dt \quad 2W\,dW + \frac{1}{2}\cdot 2\,dt$$

$$dW^2(t) = 2W(t)\,dW(t) + dt$$

$$\int_a^b dW^2(t) = 2\int_a^b W(t)\,dW(t) + \int_a^b dt$$

$$\int_a^b dW^2(t) = 2\int_a^b W(t)\,dW(t) + \int_a^b dt$$

$$W^2(t)\Big|_a^b = 2\underbrace{\int_a^b W(t)\,dW(t)} + b-a$$

$$\boxed{\int_a^b W(t)\,dW(t) = \frac{1}{2}\left[W^2(b) - W^2(a) - (b-a)\right]}$$

Example 8.10 Evaluate $\int_a^b t\,dW(t)$

$$X(t) = \int_a^t \tau\,dW(\tau) \Rightarrow dX(t) = t\,dW(t)$$

Let $F(w,t) = tw$

$$dF(W,t) = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial W}dW + \frac{1}{2}\frac{\partial^2 F}{\partial W^2}(dW)^2$$

$$d(tW) = W\,dt + t\,dW + O(dt)$$

$$\int_a^b d(tW(t)) = \int_a^b W(t)\,dt + \int_a^b t\,dW(t)$$

$$tW(t)\Big|_a^b = \int_a^b W(t)\,dt + \underbrace{\int_a^b t\,dW(t)}$$

$$\boxed{\int_a^b t\,dW(t) = bW(b) - aW(a) - \int_a^b W(t)\,dt}$$

Example 8.11 Solve the SDE (geometric Brownian motion)

$$\boxed{dX(t) = rX(t)\,dt + cX(t)\,dW(t), \quad X(0) = x_0}$$

Let $F(x,t) = \ln|x|$

$$dF(X,t) = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial x}dX + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\underbrace{(dX)^2}_{c^2X^2\underbrace{(dW)^2}_{dt}}$$

$$= 0\,dt + \frac{1}{X}dX + \frac{1}{2}c^2X^2\,dt\left(-\frac{1}{X^2}\right)$$

$$d(\ln|X(t)|) = \frac{1}{X(t)}(rX(t)\,dt + cX(t)\,dW(t))$$

$$-\frac{1}{2}\frac{1}{X^2}\cdot c^2X^2\,dt$$

$$d(\ln|X(t)|) = r\,dt + c\,dW(t) - \frac{c^2}{2}dt$$

$$= \left(r - \frac{c^2}{2}\right)dt + c\,dW(t)$$

$$\int_0^t d(\ln|X(\tau)|) = \int_0^t \left(r - \frac{c^2}{2}\right)d\tau + \int_0^t c\,dW(\tau)$$

$$\ln\frac{|X(t)|}{|X(0)|} = \left(r - \frac{c^2}{2}\right)t + cW(t).$$

$$\boxed{X(t) = X(0)e^{\left(r - \frac{c^2}{2}\right)t + cW(t)} > 0}$$

$$X(t) = X(0) + \int_0^t rX(\tau)\,d\tau + \int_0^t cX(\tau)\,dW(\tau)$$

$$E(X(t)) = X(0) + \int_0^t rE(X(\tau))\,d\tau$$

Mean: $\frac{dE(X(t))}{dt} = r\,E(X(t))$, $E(X(0)) = X(0) = x_0$

$$\boxed{\text{Mean: } \mu_X(t) = E(X(t)) = x_0 e^{rt}}$$

$$dX(t) = rX(t)\,dt + \underbrace{c\,X(t)\,dW(t)}$$

$F(x,t) = x^2$

$$dF = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial x}dX + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\underbrace{(dX)^2}_{c^2X^2(dW)^2}$$

$$= 0 + 2x\,dX + \frac{1}{2}\cdot 2\cdot c^2X^2\,dt$$

$$= 2X(rX\,\underline{dt} + cX\,dW) + c^2X^2\,\underline{dt}$$

$$dX^2(t) = (2r + c^2)X^2\,dt + \underbrace{2cX^2\,dW}_{E(dW)=0}$$

$$\frac{dE(X^2(t))}{dt} = (2r + c^2)E(X^2(t)), \quad E(X^2(0)) = x_0^2$$

$$\boxed{E(X^2(t)) = x_0^2 e^{(2r+c^2)t}}$$

Variance: $\sigma_X^2(t) = E(X^2(t)) - [E(X(t))]^2$

$$= x_0^2 e^{(2r+c^2)t} - x_0^2 e^{2rt}$$

$$\boxed{\text{Variance: } \sigma_X^2(t) = x_0^2 e^{2rt}\left(e^{c^2t} - 1\right)}$$

What happens to the mean and variance as $t \to \infty$

Geometric Brownian Motion

$$dX(t) = rX(t)\,dt + cX(t)\,dW(t)$$

$$X(0) = x_0$$

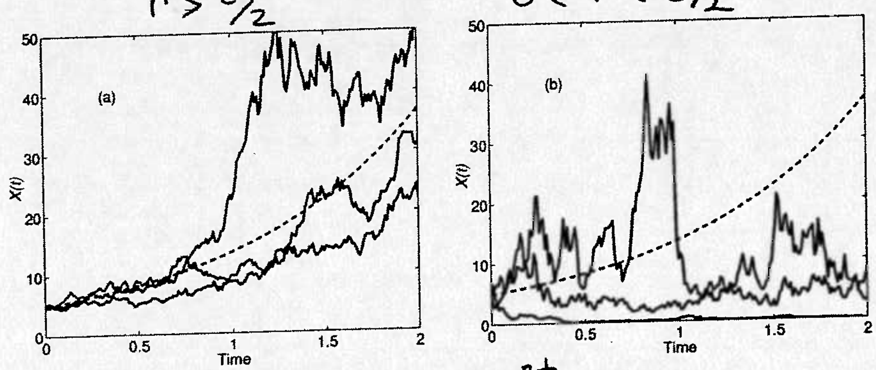
$$X(t) = X(0)\,e^{\left[\left(r - \frac{c^2}{2}\right)t + cW(t)\right]}$$

$$\begin{cases} \text{Prob}\left\{\lim\limits_{t\to\infty} X(t) = 0\right\} = 1, & \text{if } r < \frac{c^2}{2} \\ \text{Prob}\left\{\lim\limits_{t\to\infty} X(t) = \infty\right\} = 1, & \text{if } r > \frac{c^2}{2} \\ \text{(Oksendal, 2000)} & \end{cases}$$

Relation between Itô and Stratonovich SDEs

see p. 386

$r > c^2/2$ $\qquad\qquad$ $0 < r < c^2/2$



$$\mu_X(t) = 5e^{rt}$$

FIGURE 8.2: Three sample paths of the stochastic exponential growth model (8.28) with the deterministic solution, $X(0)\exp(rt)$, where $X(0) = 5$. In (a), $r = 1$ and $c = 0.5$. In (b), $r = 1$ and $c = 2$.

Relate FKDE for pdf $p(x,t)$ of $X(t)$ to the Itô SDE.

$$X(t): \quad dX(t) = \alpha(X(t),t)\,dt + \beta(X(t),t)\,dW(t)$$ sample paths

$$p(x,t): \quad \frac{\partial p}{\partial t} = -\frac{\partial(\alpha(x,t)p)}{\partial x} + \frac{1}{2}\frac{\partial^2(\beta^2(x,t)p)}{\partial x^2}$$ p.d.f

$\alpha(x,t) =$ infinitesimal mean
$\beta^2(x,t) =$ infinitesimal variance

Itô diffusion process

Example: Simple birth and death process

Birth rate λx
Death rate μx
Mean growth rate $\lambda x - \mu x$
Variance in growth rate $\lambda x + \mu x$

$$dX(t) = (\lambda-\mu)X(t)\,dt + \sqrt{(\lambda+\mu)X(t)}\,dW(t)$$

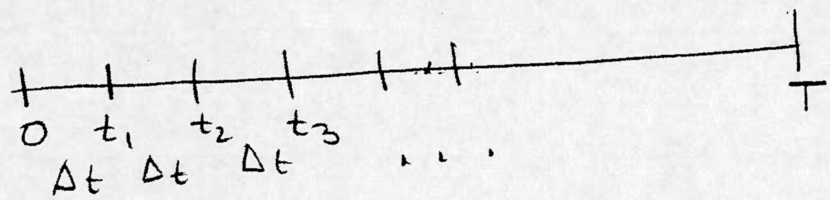
$$\frac{\partial p}{\partial t} = -\frac{\partial[(\lambda-\mu)xp]}{\partial x} + \frac{1}{2}\frac{\partial^2[(\lambda+\mu)xp]}{\partial x^2}$$

4 8.10 Numerical Methods for SDEs

Most SDE we cannot solve, so we must use a numerical method to approximate sample paths:

$$\boxed{dX(t) = \alpha(X(t), t)\,dt + \beta(X(t), t)\,dW(t)}$$

We will discuss only one method: Euler-Maruyama method - first order method.



$$\Delta t = t_{i+1} - t_i = \frac{T}{k} \qquad \Delta W(t_i) = W(t_{i+1}) - W(t_i) = W_{i+1} - W_i$$

$X(t_i)$ = exact

X_i = approximation

$$\boxed{X_{i+1} = X_i + \alpha(X_i, t_i)\,\Delta t + \beta(X_i, t_i)\,\Delta W_i}$$

$i = 0, 1, 2, \ldots, k$

$\Delta W_i \sim N(0, \Delta t)$ Let $\eta \sim N(0,1)$

then $\boxed{\eta\sqrt{\Delta t} \sim N(0, \Delta t)}$

$$X_{i+1} = X_i + \alpha(X_i, t_i)\Delta t + \beta(X_i, t_i)\eta_i\sqrt{\Delta t}$$

Where η_i is a standard normal random number.

8.11 Example Drug Kinetics

$C(t)$ = concentration of a drug in the body, time t

$$dC(t) = \underbrace{-k\,C(t)\,dt}_{\text{mean rate of drug cleared from the body}} + \underbrace{\sigma\,C(t)\,dW(t)}_{\text{standard deviation in rate of clearance}},\quad C(0) = C_0$$

Similar to geometric Brownian motion except $r = -k$ and $c = \sigma$.

Solution: $C(t) = C(0)\,e^{\left[-\left(k+\frac{\sigma^2}{2}\right)t + \sigma W(t)\right]}$

$$\text{Prob}\left\{\lim_{t\to\infty} C(t) = 0\right\} = 1$$

Continuous drug dosing at a rate α to maintain a constant level C_S: (16)

$$\boxed{dC(t) = \alpha(C_S - C(t))dt + \sigma C(t) dW(t)}$$

$$\frac{dE(C(t))}{dt} = \alpha\left(C_S - E(C(t))\right) \qquad C(0) = C_0$$

$$\mu_C(t) = E(C(t)) = C_S + e^{-\alpha t}(C_0 - C_S) \to \underline{\underline{C_S}}$$

$$\sigma_C^2(t) \to \begin{cases} \dfrac{\sigma^2 C_S^2}{2\alpha - \sigma^2}, & \text{if } \alpha > \dfrac{\sigma^2}{2} \\ \infty, & \text{if } \alpha < \dfrac{\sigma^2}{2} \end{cases}$$

Program:

$$\begin{cases} \text{for } t = 1 : time/dt \\ C(t+1) = C(t) + \alpha * (C_S - C(t)) * dt + \sigma * C(t) * \underbrace{\sqrt{\Delta t}\ randn} \\ \text{end} \end{cases}$$

$\alpha = 3$, $\sigma = 1$, $C_S = 10$

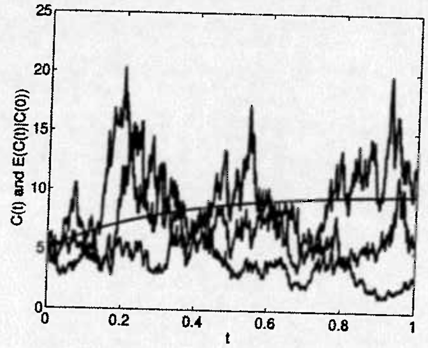


FIGURE 8.5: Three sample paths of $C(t)$ and the mean $E(C(t)|C(0))$ (smooth curve) for the drug kinetics example with parameter values and initial condition $\alpha = 3$, $\sigma = 1$, $C_s = 10$, and $C(0) = 5$ for $t \in [0, 1]$, $2\alpha > \sigma^2$.