Chapter 6: Continuous-Time Birth and Death Processes:

The generator matrix Q for a general birth and death process is either

$$Q = \begin{pmatrix} -\lambda_0 & \mu_1 & 0 & 0 & \cdots \\ \lambda_0 & -\lambda_1 - \mu_1 & \mu_2 & 0 & \cdots \\ 0 & \lambda_1 & -\lambda_2 - \mu_2 & \mu_3 & \cdots \\ 0 & 0 & \lambda_2 & -\lambda_3 - \mu_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix},$$

or for a finite state space,

$$Q = \begin{pmatrix} -\lambda_0 & \mu_1 & 0 & \cdots & 0 \\ \lambda_0 & -\lambda_1 - \mu_1 & \mu_2 & \cdots & 0 \\ 0 & \lambda_1 & -\lambda_2 - \mu_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \mu_N \\ 0 & 0 & 0 & \cdots & -\mu_N \end{pmatrix}.$$

The simple birth, simple death, simple birth and death, and simple birth, death, and immigration processes are linear in the rates, $\lambda_i = \lambda i + \nu$ and $\mu_i = \mu i$.

Applying the forward Kolmogorov differential equations, $\dfrac{dp}{dt} = Qp$, first-order partial differential equations for the p.g.f. and m.g.f. can be derived. Applying the method of characteristics to the first-order partial differential equations, explicit expressions can be found for the p.g.f. and m.g.f. of these processes. Denote the p.g.f. for these simple process as follows (note change in notation z and θ because t is time):

$$\mathcal{P}(z,t) = \sum_{i=0}^{\infty} p_i(t) z^i$$

$$M(\theta,t) = \sum_{i=0}^{\infty} p_i(t) e^{i\theta} = \mathcal{P}(e^\theta, t)$$

Simple Birth, Death, and Immigration Process: Let $X(0) = N$. The infinitesimal transition probabilities are

$$\begin{aligned} p_{i+j,i}(\Delta t) &= \text{Prob}\{\Delta X(t) = j | X(t) = i\} \\ &= \begin{cases} \mu i \, \Delta t + o(\Delta t), & j = -1 \\ (\nu + \lambda i)\Delta t + o(\Delta t), & j = 1 \\ 1 - [\nu + (\lambda + \mu)i] \, \Delta t + o(\Delta t), & j = 0 \\ o(\Delta t), & j \neq -1, \ 0, \ 1. \end{cases} \end{aligned}$$

The forward Kolmogorov differential equations are

$$\begin{aligned} \frac{dp_i}{dt} &= [\lambda(i-1) + \nu] p_{i-1} + \mu(i+1) p_{i+1} - (\lambda i + \mu i + \nu) p_i \\ \frac{dp_0}{dt} &= -\nu p_0 + \mu p_1 \end{aligned}$$

for $i = 1, 2, \ldots$ with initial condition $X(0) = N$. Applying the generating function technique, it follows that the m.g.f. $M(\theta, t)$ is a solution of the following first-order partial differential equation

$$\frac{\partial M}{\partial t} = \left[\lambda(e^\theta - 1) + \mu(e^{-\theta} - 1)\right] \frac{\partial M}{\partial \theta} + \nu(e^\theta - 1) M$$

with initial condition $M(\theta,0)=e^{N\theta}$. The preceding differential equation is first-order because the rates are linear. The m.g.f. is the solution of this first-order partial differential equation,

$$M(\theta,t)=\frac{(\lambda-\mu)^{\nu/\lambda}\left[\mu(e^{(\lambda-\mu)t}-1)-e^{\theta}(\mu e^{(\lambda-\mu)t}-\lambda)\right]^N}{\left[(\lambda e^{(\lambda-\mu)t}-\mu)-\lambda(e^{(\lambda-\mu)t}-1)e^{\theta}\right]^{N+\nu/\lambda}}.$$

The moments $E(X^n(t))$ of the probability distribution $X(t)$ can be found by differentiating the m.g.f. with respect to θ and evaluating at $\theta=0$:

$$E(X^n(t))=\left.\frac{\partial^n M(\theta,t)}{\partial\theta^n}\right|_{\theta=0}.$$

Table 1: Mean, variance, and p.g.f. for the simple birth, simple death, and simple birth and death processes, where $X(0)=N$ and $\rho=e^{(\lambda-\mu)t}$, $\lambda\neq\mu$

	Simple Birth	Simple Death	Simple Birth and Death
$m(t)$	$Ne^{\lambda t}$	$Ne^{-\mu t}$	$Ne^{(\lambda-\mu)t}$
$\sigma^2(t)$	$Ne^{2\lambda t}(1-e^{-\lambda t})$	$Ne^{-\mu t}(1-e^{-\mu t})$	$N\frac{\lambda+\mu}{\lambda-\mu}\rho(\rho-1)$
$\mathcal{P}(z,t)$	$\frac{(pz)^N}{(1-z(1-p))^N}$ Negative binomial $p=e^{-\lambda t}$	$(1-p+pz)^N$ Binomial $b(N,p)$ $p=e^{-\mu t}$	$\left(\frac{\rho^{-1}(\lambda z-\mu)-\mu(z-1)}{\rho^{-1}(\lambda z-\mu)-\lambda(z-1)}\right)^N$

Table 2: Mean, variance, and p.g.f. for the simple birth and death with immigration process, where $X(0)=N$ and $\rho=e^{(\lambda-\mu)t}$, $\lambda\neq\mu$

	Simple Birth and Death with Immigration
$m(t)$	$\frac{\rho[N(\lambda-\mu)+\nu]-\nu}{\lambda-\mu}$
$\sigma^2(t)$	$N\frac{(\lambda^2-\mu^2)\rho[\rho-1]}{(\lambda-\mu)^2}+\nu\frac{\mu+\rho(\lambda\rho-\mu-\lambda)}{(\lambda-\mu)^2}$
$\mathcal{P}(z,t)$	$\frac{(\lambda-\mu)^{\nu/\lambda}\left[\mu(\rho-1)-z(\mu\rho-\lambda)\right]^N}{\left[\lambda\rho-\mu-\lambda(\rho-1)z\right]^{N+\nu/\lambda}}$

For each of the preceding simple birth and death processes, the probability of extinction $p_0(t)$ can be found by evaluating the p.g.f. at $z=0$, $\mathcal{P}(0,t)$, and the probability of ultimate extinction by taking the limit: $\lim_{t\to\infty}p_0(t)$. Except for the stochastic process with immigration, the other birth and death processes either hit zero or approach infinity. For example, given $X(0) = N$, for the simple birth process, $p_0(t) = 0$, simple death process $p_0(t) = (1 - e^{-\mu t})^N$ and for the simple birth and death process:

$$\lim_{t\to\infty} p_0(t) = \begin{cases} \left(\dfrac{\mu}{\lambda}\right)^N, & \lambda > \mu \\ 1, & \lambda \le \mu \end{cases}$$

Logistic Growth Model: $\dfrac{dn}{dt} = rn\left(1 - \dfrac{n}{K}\right)$ What is the birth rate and what is the death rate? There are an infinite number of choices for stochastic birth and death rates that yield the same deterministic logistic growth model.

The birth and death rates should have the form

$$\lambda_i = b_1 i + b_2 i^2 > 0 \quad \text{and} \quad \mu_i = d_1 i + d_2 i^2 > 0$$

so that $\lambda_n - \mu_n = rn(1 - n/K)$.

$$\frac{dn}{dt} = (b_1 - d_1)n + (b_2 - d_2)n^2 = rn - \frac{r}{K}n^2,$$

leads to

$$b_1 - d_1 = r > 0 \quad \text{and} \quad \frac{b_1 - d_1}{d_2 - b_2} = K > 0.$$

Define the birth and death rates, λ_i and μ_i, as follows:

(a) $\lambda_i = i$ and $\mu_i = \dfrac{i^2}{10}$, $i = 0, 1, 2, \ldots$

(b) $\lambda_i = \begin{cases} i - \dfrac{i^2}{20}, & i = 0, 1, \ldots, 20 \\ 0, & i > 20 \end{cases}$ and $\mu_i = \dfrac{i^2}{20}$, $i = 0, 1, 2, \ldots$

In both cases the deterministic model is

$$\frac{dn}{dt} = n\left(1 - \frac{n}{10}\right),$$

where $r = 1$ and $K = 10$. In the deterministic model, solutions approach the carrying capacity $K = 10$. Three sample paths for models (a) and (b) are graphed in Figure 1.

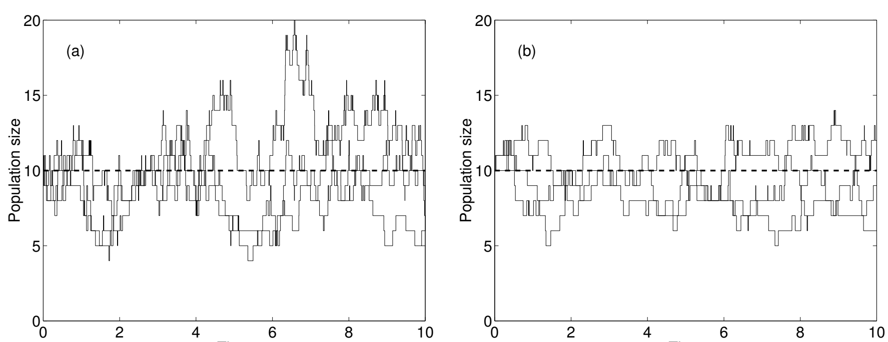

Figure 1: Three sample paths of the stochastic logistic model for cases (a) and (b) with $X(0) = 10$.

For these stochastic logistic models,

$$\lim_{t\to\infty} p_0(t) = 1.$$

For a population of finite size N, we will show that the mean time to extinction is a solution of the following matrix equation,

$$\boxed{\tau\tilde{Q} = -\mathbf{1}}$$

where $\tilde{Q}$ is the truncated generator matrix without state zero, $\tau = (\tau_1, \ldots, \tau_N)$, τ_k is the mean time until extinction from state k, $\tau = -\mathbf{1}\tilde{Q}^{-1}$. Let

$$\lambda_i = \begin{cases} i - \dfrac{i^2}{N}, & i = 0, 1, \ldots, N \\ 0, & i > N \end{cases} \quad \text{and} \quad \mu_i = \frac{i^2}{N}, \quad i = 0, 1, 2, \ldots, N.$$

The intrinsic growth rate $r = 1$ and the carrying capacity $K = N/2$. The expected time to extinction can be calculated for $X(0) = m$, $m = 1, 2, \ldots, N$ by solving $\tau\tilde{Q} = -\mathbf{1}$. When $N = 10$, the carrying capacity is $K = 5$, and when $N = 20$, the carrying capacity is $K = 10$. The time units depend on the particular problem. If the time is measured in days, then for the first example, extinction occurs, on the average, in less than one year but for the second example, the mean time to extinction is over 300 years. Hence, for even larger values of K, the mean time to extinction will take much longer; the convergence of $p_0(t)$ to 1 is slow.

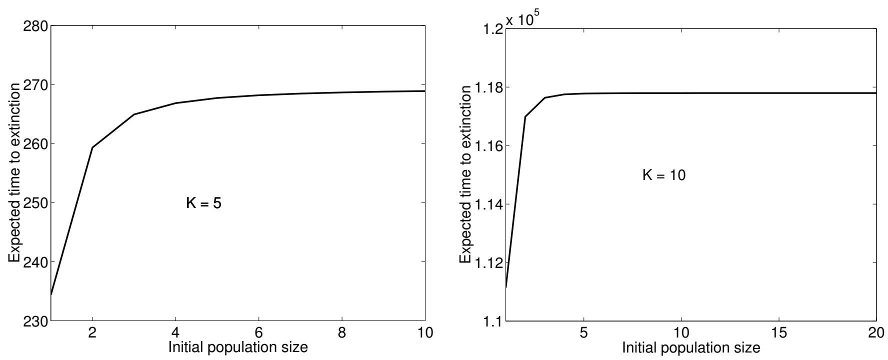

Figure 2: Expected time until extinction in the stochastic logistic model with $K = 5$ and $K = 10$.