Probability Distributions

The probability mass function (p.m.f.) or probability density functions (p.d.f.), mean, μ, variance, σ^2, and moment generating function (m.g.f.) $M(t)$ for some well-known discrete and continuous probability distributions.

Discrete Distributions

Discrete Uniform: $f(x) = \begin{cases} \frac{1}{n}, & x = 1, 2, \ldots, n, \\ 0, & \text{otherwise.} \end{cases}$

$\mu = \frac{1+n}{2}$, $\sigma^2 = \frac{n^2-1}{12}$, $M(t) = \frac{e^{(n+1)t} - e^t}{n(e^t - 1)}$, $t \neq 0$.

Geometric: $f(x) = \begin{cases} p(1-p)^x, & x = 0, 1, 2, \ldots, \\ 0, & \text{otherwise,} \end{cases}$ where $0 < p < 1$.

$\mu = \frac{1-p}{p}$, $\sigma^2 = \frac{1-p}{p^2}$, $M(t) = \frac{p}{1-(1-p)e^t}$.

Binomial $b(n,p)$: $f(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x = 0, 1, 2, \ldots, n, \\ 0, & \text{otherwise,} \end{cases}$ where n is a positive integer and $0 < p < 1$. The notation $\binom{n}{x} = \frac{n!}{x!(n-x)!}$.

$\mu = np$, $\sigma^2 = np(1-p)$, $M(t) = (1 - p + pe^t)^n$.

Negative Binomial: $f(x) = \begin{cases} \binom{x+n-1}{n-1} p^n (1-p)^x, & x = 0, 1, 2, \ldots, \\ 0, & \text{otherwise,} \end{cases}$ where n is a positive integer and $0 < p < 1$.

$\mu = \frac{n(1-p)}{p}$, $\sigma^2 = \frac{n(1-p)}{p^2}$, $M(t) = \frac{p^n}{[1-(1-p)e^t]^n}$.

Poisson, $Po(\lambda)$: $f(x) = \begin{cases} \frac{\lambda^x e^{-\lambda}}{x!}, & x = 0, 1, 2, \ldots, \\ 0, & \text{otherwise,} \end{cases}$ where λ is a positive constant.

$\mu = \lambda$, $\sigma^2 = \lambda$, $M(t) = e^{\lambda(e^t - 1)}$.

Continuous Distributions

Uniform $U(a,b)$: $f(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b, \\ 0, & \text{otherwise,} \end{cases}$ where $a < b$ are constants.

$\mu = \frac{a+b}{2}$, $\sigma^2 = \frac{(b-a)^2}{12}$, $M(t) = \frac{e^{bt} - e^{at}}{t(b-a)}$, $t \neq 0$.

Gamma: $f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, & x \geq 0, \\ 0, & x < 0, \end{cases}$ where α and β are positive constants and $\Gamma(\alpha) = \int_0^\infty \frac{1}{\beta^\alpha} x^{\alpha-1} e^{-x/\beta}\, dx$. For a positive integer n, $\Gamma(n) = (n-1)!$.

$\mu = \alpha\beta$, $\sigma^2 = \alpha\beta^2$, $M(t) = \frac{1}{(1-\beta t)^\alpha}$, $t < 1/\beta$.

Exponential: $f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0, \end{cases}$ where λ is a positive constant.

$\mu = \frac{1}{\lambda}$, $\sigma^2 = \frac{1}{\lambda^2}$, $M(t) = \frac{\lambda}{\lambda - t}$, $t < \lambda$.

Normal, $N(\mu, \sigma^2)$: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$, $-\infty < x < \infty$, where μ and σ are constants.

$E(X) = \mu$, $Var(X) = \sigma^2$, $M(t) = e^{\mu t + \sigma^2 t^2/2}$.

The probability generating function (p.g.f.) is $\mathcal{P}_X(t) = E(t^X)$, moment generating function (m.g.f.) is $M_X(t) = E(e^{tX})$ and cumulant generating function (c.g.f.) is $K_X(t) = \ln(M_X(t))$. The mean $\mu_X = E(X)$ and variance $\sigma_X^2 = E(X^2) - E^2(X)$ computed from the p.g.f, $\mathcal{P}_X(t)$, m.g.f., $M_X(t)$ and c.g.f., $K_X(t)$:

$$\boxed{\mu_X = \mathcal{P}'_X(1) = M'_X(0) = K'_X(0)}$$

and

$$\boxed{\sigma_X^2 = \begin{cases} \mathcal{P}''_X(1) + \mathcal{P}'_X(1) - [\mathcal{P}'_X(1)]^2 \\ M''_X(0) - [M'_X(0)]^2 \\ K''_X(0) \end{cases}}$$

Continuous-time Markov Chain Example, Simple Birth Process: The per capita rate of birth is $\lambda = 1$

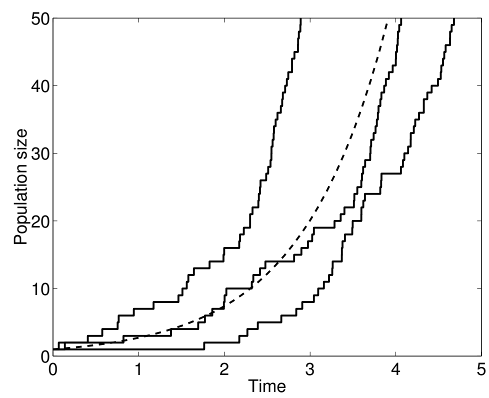

Figure 1: Three sample paths (stochastic realizations) of a simple birth process, $X(0) = 1$; $n(t) = e^t$ is the dashed curve. See Table 1.

Table 1: For two stochastic realizations, the times at which a birth occurs for a simple birth process.

Realization 1		Realization 2	
Size $X(t)$	Event Time t	Size $X(t)$	Event Time t
1	0	1	0
2	0.138	2	1.764
3	0.407	3	2.174
4	0.575	4	2.269
5	0.755	5	2.390
6	0.766	6	2.664
7	0.943	7	2.839
8	1.173	8	2.909
9	1.463	9	3.044
10	1.511	10	3.095
$\vdots$	$\vdots$	$\vdots$	$\vdots$
50	2.890	50	4.678