# Answers Written Assignment #6

1. Taking the Fourier transform of the PDE: $\frac{dU}{dt} = -\alpha^2 k U \Rightarrow U(\alpha,t) = \underline{c} e^{-\alpha^2 k t}$

Taking the Fourier transform of the IC. $U(\alpha,0) = \mathcal{F}\{2e^{-|x|}\} = \underline{\frac{4}{\alpha^2+1}}$

Thus, $U(\alpha,t) = \frac{4}{\alpha^2+1} e^{-k\alpha^2 t}$. Now, take the inverse Fourier transform of $U(\alpha,t)$

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4}{\alpha^2+1} e^{-k\alpha^2 t} e^{-i\alpha x}\, d\alpha = \frac{2}{\pi}\underbrace{\int_{-\infty}^{\infty} \frac{e^{-k\alpha^2 t}}{\alpha^2+1}}_{\text{even}} [\underbrace{\cos(\alpha x)}_{\text{even}} - i\underbrace{\sin(\alpha x)}_{\text{odd}}]\, d\alpha$$

$$\boxed{u(x,t) = \frac{4}{\pi}\int_0^{\infty} \frac{e^{-k\alpha^2 t}}{\alpha^2+1}\cos(\alpha x)\, d\alpha}$$

2. Apply the Fourier sine transform: $\frac{dU}{dt} = -\alpha^2 k U + \alpha \cancel{u(0,t)}^{0} \Rightarrow \frac{dU}{dt} = -\alpha^2 k U \Rightarrow \underline{U(\alpha,t) = c e^{-\alpha^2 k t}}$

$U(\alpha,0) = \mathcal{F}_S\{e^{-2x}\} = \underline{\frac{\alpha}{\alpha^2+4}} \Rightarrow U(\alpha,t) = \frac{\alpha}{\alpha^2+4} e^{-\alpha^2 k t}$

$\mathcal{F}_S^{-1}\{U(\alpha,t)\} = \mathcal{F}_S^{-1}\left\{\frac{\alpha}{\alpha^2+4} e^{-\alpha^2 k t}\right\} \Rightarrow \boxed{u(x,t) = \frac{2}{\pi}\int_0^{\infty} \frac{\alpha}{\alpha^2+4} e^{-\alpha^2 k t}\sin(\alpha x)\, d\alpha}$

3. Apply the Fourier cosine transform: $\frac{d^2U}{dt^2} = -\alpha^2 U - u_x(0,t) \Rightarrow \frac{d^2U}{dt^2} + \alpha^2 U = 0$

$\underline{U(\alpha,t) = c_1\cos(\alpha t) + c_2\sin(\alpha t)}$, $U(\alpha,0) = \mathcal{F}_c\{3e^{-x}\}$ $\left.\frac{dU}{dt}\right|_{t=0} = \mathcal{F}_c\{0\}$

$U(\alpha,0) = \frac{3}{\alpha^2+1}$, $\frac{dU}{dt} = 0 \Rightarrow U(\alpha,t) = \frac{3}{\alpha^2+1}\cos(\alpha t)$.

$\mathcal{F}_c^{-1}\{U(\alpha,t)\} = \mathcal{F}_c^{-1}\left\{\frac{3}{\alpha^2+1}\cos(\alpha t)\right\} \Rightarrow \boxed{u(x,t) = \frac{2}{\pi}\int_0^{\infty} \frac{3}{\alpha^2+1}\cos(\alpha t)\cos(\alpha x)\, d\alpha}$

4. Apply the Fourier sine transform: $\frac{d^2U}{dt^2} = -\alpha^2 U + \alpha \cancel{u(0,t)}^{0} \Rightarrow \frac{d^2U}{dt^2} + \alpha^2 U = 0$

$\underline{U(\alpha,t) = c_1\cos(\alpha t) + c_2\sin(\alpha t)}$ $U(\alpha,0) = \mathcal{F}_S\{5xe^{-x}\} = \frac{10\alpha}{(\alpha^2+1)^2}$

$\left.\frac{dU}{dt}\right|_{t=0} = \mathcal{F}_S\{0\} = 0$

$U(\alpha,t) = \frac{10}{(\alpha^2+1)^2}\cos(\alpha t)$

$\mathcal{F}_S^{-1}\{U(\alpha,t)\} = \mathcal{F}_S^{-1}\left\{\frac{10}{(\alpha^2+1)^2}\cos(\alpha t)\right\}$

$$\boxed{u(x,t) = \frac{2}{\pi}\int_0^{\pi} \frac{10}{(\alpha^2+1)^2}\cos(\alpha t)\sin(\alpha x)\, d\alpha}$$