Practice Problems Answers

1. $u(x,t) = v(x,t) + \psi(x)$ ① $v_t = v_{xx}$ $0<x<1$, $t>0$ ② $\psi'' + 3x^2 - 1 = 0$

① + ② $v(0,t)=0$, $v(1,t)=1$, $t>0$ $\psi(0)=2$, $\psi(1)=1$

$v(x,t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi)^2 t} \sin(n\pi x)$ $\psi' = -x^3 + x + c_1$

General solution $\psi = -\frac{x^4}{4} + \frac{x^2}{2} + c_1 x + c_2$

$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi)^2 t} \sin(n\pi x) - \frac{x^4}{4} + \frac{x^2}{2} - \frac{5}{4}x + 2$ $\psi(0) = 2 = c_2$

$u(x,0) = 2 = \sum A_n \sin(n\pi x) + \cancel{2} - \frac{x^4}{4} + \frac{x^2}{2} - \frac{5}{4}x + \cancel{2}$ $\psi(1) = -\frac{1}{4} + \frac{1}{2} + c_1 + 2 = 1$

$c_1 = -\frac{5}{4}$

$\sum_{n=1} A_n \sin(n\pi x) = \frac{x^4}{4} - \frac{x^2}{2} + \frac{5}{4}x$ Steady State: $\boxed{\psi(x) = -\frac{x^4}{4} + \frac{x^2}{2} - \frac{5}{4}x + 2}$

$\Rightarrow$ $\boxed{A_n = \frac{2}{1}\int_0^1 \left(\frac{x^4}{4} - \frac{x^2}{2} + \frac{5}{4}x\right) \sin(n\pi x)\,dx}$

$\Rightarrow$ $u(x,t) = \boxed{\psi(x) + \sum_{n=1}^{\infty} A_n e^{-(n\pi)^2 t} \sin(n\pi x)}$

2. $\boxed{u(r,\theta) = 5 - 7\left(\frac{r}{5}\right)^4 \sin(4\theta)}$ maximum $5+7 = 12$

minimum $5-7 = -2$

3. (a) Fourier Sine transform

$\mathcal{F}_S\{u_t\} = \mathcal{F}_S\{u_{xx}\} \Rightarrow \frac{dU}{dt} = -\alpha^2 U + \alpha \cancel{u(0,t)} \Rightarrow U(\alpha,t) = ce^{-\alpha^2 t}$

$U(\alpha,0) = \mathcal{F}_S\{xe^{-0.2x}\} = \frac{2(0.2)\alpha}{(\alpha^2 + (.2)^2)^2} = c$

$U(\alpha,t) = \frac{0.4\alpha}{(\alpha^2 + .04)^2} e^{-\alpha^2 t}$, $\boxed{u(x,t) = \mathcal{F}_S^{-1}\{U(\alpha,t)\} = \frac{2}{\pi}\int_0^\infty \frac{0.4\alpha}{(\alpha^2 + .04)^2} e^{-\alpha^2 t} \sin(\alpha x)\,d\alpha}$

(b) Double Fourier Sine series $\lambda_m = \left(\frac{m\pi}{1}\right)^2$ $\mu_n = \left(\frac{n\pi}{1}\right)^2$

$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn} e^{-k[(n\pi)^2 + (m\pi)^2]t} \sin(m\pi x)\sin(n\pi x)$

$u(x,y,0) = 5\sin(2\pi x)\sin(3\pi y) \Rightarrow$ $A_{23} = 5$ $A_{mn} = 0$ otherwise

$\boxed{u(x,y,t) = 5e^{-k[(4\pi)^2 + (3\pi)^2]t} \sin(2\pi x)\sin(3\pi y)}$

(c) $\boxed{u(x,y,t) = 3}$

(d) $u(x,t) = e^{-4t}\left[\sum_{n=1}^{\infty} A_n e^{-\left(\frac{2n-1}{2}\pi\right)^2 t} \sin\left(\frac{2n-1}{2}\pi x\right)\right]$

$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{2n-1}{2}\pi x\right) = 2\sin\left(\frac{3x}{2}\right)$, $A_1 = 2$

$\boxed{u(x,t) = e^{-4t}\left[2e^{-\left(\frac{3}{2}\right)^2 kt} \sin\left(\frac{3x}{2}\right)\right] = 2e^{-4t - \frac{9}{4}kt} \cdot \sin\left(\frac{3x}{2}\right)}$

(e) Fourier Cosine $\frac{d^2U}{dt^2} = -\alpha^2 U - \cancel{u_x(0,t)} = \alpha^2 U$, $U = c_1\cos(\alpha t) + c_2\sin(\alpha t)$

$\mathcal{F}_C\{u(x,0)\} = U(\alpha,0) = 0$

$\mathcal{F}_C\left\{\frac{\partial u}{\partial t}\Big|_{t=0}\right\} = \frac{dU}{dt}\Big| = \frac{2}{\pi}\int_0^1 2\cos(\alpha x)\,dx = \frac{4}{\pi\alpha}\sin(\alpha x)\Big|_0^1 = \frac{4}{\pi\alpha}\sin(\alpha)$ over $\rightarrow$

$V(\alpha,0) = c_1 = 0 \qquad \left.\frac{dV}{dt}\right|_{t=0} = c_2\alpha = \frac{4}{\pi\alpha}\sin(\alpha) \Rightarrow c_2 = \frac{4}{\pi\alpha^2}\sin(\alpha)$

$V(\alpha,t) = \frac{4}{\pi\alpha^2}\sin(\alpha)\cdot\sin(\alpha t)$

$\mathcal{F}_c^{-1}\{V(\alpha,t)\} = \mathcal{F}_c^{-1}\left\{\frac{4}{\pi\alpha^2}\sin(\alpha)\sin(\alpha t)\right\}$

$$\boxed{u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{4}{\pi\alpha^2}\sin(\alpha)\sin(\alpha t)\cos(\alpha x)\,d\alpha}$$