Asymptotic Expansions for Decaying Solutions of Dissipative Differential Equations

based on paper https://arxiv.org/abs/2009.06769

Joint work with Dat Cao, Minnesota State University, Mankato
and Thinh Kieu, University of North Georgia, Gainesville Campus

Luan T. Hoang

Department of Mathematics and Statistics, Texas Tech University

Joint Mathematics Meetings AMS-MAA
Special Session on Geophysical Fluid Dynamics, Turbulence, and Data Assimilation: A Rigorous and Computational Study
January 6–9, 2021.

Outline

1. Introduction

The Navier–Stokes equations

The Eulerian description turns out to be simpler for deriving the set of equations that govern the fluid flows. They are called the Navier–Stokes equations (NSE),

$$\begin{cases} u_t - \nu\Delta u + (u\cdot\nabla)u = -\nabla p + f, \\ \operatorname{div} u = 0. \end{cases}$$

where $\nu > 0$ is the kinematic viscosity, and the unknowns are the velocity $u(x,t)$ and pressure $p(x,t)$.
Initial condition $u(x,0) = u_0(x)$, where u_0 is a given initial vector field.

Foias–Saut asymptotic expansions

Functional form of NSE (after scaling to have $\nu = 1$):

$$u_t + Au + B(u,u) = f, \quad u(0) = u_0.$$

• If $f = const. \neq 0$, turbulence.
• If $f = 0$ or $f = f(t) \to 0$ as $t \to \infty$, turbulence for short time, then the flows settle (to zero) eventually.
• Consider $f = 0$. Foias–Saut (1987) proved that any Leray–Hopf weak solution $u(t)$ has an asymptotic expansion,

$$u(t) \sim \sum_{n=1}^{\infty} q_n(t) e^{-\mu_n t},$$

where $q_j(t)$'s are polynomials in t with values in functional spaces.
• In fact, there is a smallest n_0 such that $q_{n_0} \neq 0$ is independent of t, is an eigenfunction of the Stokes operator A.

Asymptotic expansions

Let $(X, \|\cdot\|)$ be a normed space and $(\alpha_n)_{n=1}^\infty$ be a sequence of strictly increasing non-negative numbers. A function $f : [T,\infty) \to X$, for some $T \in \mathbb{R}$, is said to have an asymptotic expansion

$$f(t) \sim \sum_{n=1}^{\infty} f_n(t) e^{-\alpha_n t} \quad \text{in } X,$$

where $f_n(t)$ is an X-valued polynomial, if one has, for any $N \geq 1$, that

$$\left\| f(t) - \sum_{n=1}^{N} f_n(t) e^{-\alpha_n t} \right\| = \mathcal{O}(e^{-(\alpha_N + \varepsilon_N)t}) \quad \text{as } t \to \infty,$$

for some $\varepsilon_N > 0$.

Exponential decaying rates

• Denote the spectrum of the Stokes operator A by $\{\Lambda_k : k \in \mathbb{N}\}$, where Λ_k's are positive, strictly increasing to infinity.
• Let $\mathcal{S}$ be the additive semigroup generated by Λ_k's, that is,

$$\mathcal{S} = \Big\{ \sum_{j=1}^{N} \Lambda_{k_j} : N, k_1, \ldots, k_N \in \mathbb{N} \Big\}.$$

• We arrange the set $\mathcal{S}$ as a sequence $(\mu_n)_{n=1}^{\infty}$ of positive, strictly increasing numbers. Clearly,

$$\lim_{n\to\infty} \mu_n = \infty,$$

$$\mu_n + \mu_k \in \mathcal{S} \quad \forall n, k \in \mathbb{N}.$$

Other NSE and PDE results

- H.–Martinez (2017, 2018) prove that the Foias–Saut expansion holds in Gevrey spaces with non-potential force

$$u_t + Au + B(u,u) = f(t) \sim \sum_{n=1}^{\infty} f_n(t) e^{-\gamma_n t}.$$

- Cao–H. (2020)

$$u_t + Au + B(u,u) = f(t) \sim \sum_{n=1}^{\infty} \chi_n t^{-\gamma_n}.$$

- H.–Titi (2020): Rotating fluids

$$u_t - \nu \Delta u + (u \cdot \nabla) u + R e_3 \times u = -\nabla p.$$

- Dissipative wave equations: Shi (2000)
- Navier–Stokes–Boussinesq system: Biswas–H.–Martinez (in preparation)

ODE results

A. With analytic nonlinear terms, no forcing.

$$y' + Ay = F(y).$$

- Normal forms: Poincaré, Dulac, Lyapunov (first method), Bruno.
- Power geometry: Bruno (1960s–present).
- Foias–Saut approach: Minea (1998).

B. Lagrangian trajectories. H. (2020):

$$y' = u(y, t).$$

C. With forcing.

$$y' + Ay = F(y) + f(t).$$

- Cao–H. (2020).

$$f(t) = \sum t^{-\mu}, (\ln t)^r, (\ln \ln t)^r (\ln \ln \ln t)^r, \ldots.$$

2. Main result

Consider ODE in $\mathbb{R}^d$:

$$\frac{\mathrm{d}y}{\mathrm{d}t} + Ay = F(y), \quad t > 0,$$

where A is a $d \times d$ constant (real) matrix, and F is a vector field on $\mathbb{R}^d$.

Assumption

Matrix A is a diagonalizable with positive eigenvalues.

• The spectrum $\sigma(A)$ of matrix A consists of eigenvalues Λ_k's, for $1 \le k \le d$, which are positive and increasing in k.

• Then there exists an invertible matrix S such that

$$A = S^{-1} A_0 S, \text{ where } A_0 = \mathrm{diag}[\Lambda_1, \Lambda_2, \dots, \Lambda_d].$$

• Denote the distinct eigenvalues of A by λ_j's that are strictly increasing in j, i.e.,

$$0 < \lambda_1 = \Lambda_1 < \lambda_2 < \ldots < \lambda_{d_*} = \Lambda_d \quad \text{with } 1 \le d_* \le d.$$

Positively homogeneous functions

Definition

Suppose $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two (real) normed spaces.
A function $F : X \to Y$ is positively homogeneous of degree $\beta \geq 0$ if

$$F(tx) = t^\beta F(x) \text{ for any } x \in X \text{ and any } t > 0.$$

Define $\mathcal{H}_\beta(X, Y)$ to be the set of positively homogeneous functions of order β from X to Y, and denote $\mathcal{H}_\beta(X) = \mathcal{H}_\beta(X, X)$.
For a function $F \in \mathcal{H}_\beta(X, Y)$, define

$$\|F\|_{\mathcal{H}_\beta} = \sup_{\|x\|_X=1} \|F(x)\|_Y = \sup_{x \neq 0} \frac{\|F(x)\|_Y}{\|x\|_X^\beta}.$$

The following are immediate properties.

1. If $F \in \mathcal{H}_\beta(X, Y)$ with $\beta > 0$, then taking $x = 0$ and $t = 2$ gives
$$F(0) = 0.$$
If, in addition, F is bounded on the unit sphere in X, then
$$\|F\|_{\mathcal{H}_\beta} \in [0, \infty) \text{ and } \|F(x)\|_Y \le \|F\|_{\mathcal{H}_\beta} \|x\|_X^\beta \quad \forall x \in X.$$
2. The zero function (from X to Y) belongs to $\mathcal{H}_\beta(X, Y)$ for all $\beta \ge 0$, and a constant function (from X to Y) belongs to $\mathcal{H}_0(X, Y)$.
3. Each $\mathcal{H}_\beta(X, Y)$, for $\beta \ge 0$, is a linear space.
4. If $F_1 \in \mathcal{H}_{\beta_1}(X, \mathbb{R})$ and $F_2 \in \mathcal{H}_{\beta_2}(X, Y)$, then $F_1 F_2 \in \mathcal{H}_{\beta_1+\beta_2}(X, Y)$.
5. If $F : X \to Y$ is a homogeneous polynomial of degree $m \in \mathbb{Z}_+$, then $F \in \mathcal{H}_m(X, Y)$.

Assumption

The mapping $F : \mathbb{R}^d \to \mathbb{R}^d$ has the the following properties.

1. *F is locally Lipschitz on $\mathbb{R}^d$ and $F(0) = 0$.*
2. *Either (a) or (b) below is satisfied.*
   (a) There exist numbers β_k's, for $k \in \mathbb{N}$, which belong to $(1, \infty)$ and increase strictly to infinity, and functions $F_k \in \mathcal{H}_{\beta_k}(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d_0)$, for $k \in \mathbb{N}$, such that it holds, for any $N \in \mathbb{N}$, that

$$\left| F(x) - \sum_{k=1}^{N} F_k(x) \right| = \mathcal{O}(|x|^\beta) \text{ as } x \to 0, \text{ for some } \beta > \beta_N.$$

   (b) There exist $N_* \in \mathbb{N}$, strictly increasing numbers β_k's in $(1, \infty)$, and functions $F_k \in \mathcal{H}_{\beta_k}(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d_0)$, for $k = 1, 2, \ldots, N_*$, such that

$$\left| F(x) - \sum_{k=1}^{N_*} F_k(x) \right| = \mathcal{O}(|x|^\beta) \text{ as } x \to 0, \text{ for all } \beta > \beta_{N_*}.$$

Main Theorem

Theorem (Cao-H.-Kieu 2020)

Let $y(t)$ be a non-trivial, decaying solution. Then there exist polynomials $q_n\colon \mathbb{R} \to \mathbb{R}^d$ such that $y(t)$ has an asymptotic expansion

$$y(t) \sim \sum_{n=1}^{\infty} q_n(t) e^{-\mu_n t} \text{ in } \mathbb{R}^d,$$

where μ_n's are increasing strictly to infinity, and $q_n(t)$ satisfies, for any $n \geq 1$,

$$q_n' + (A - \mu_n I_d) q_n = \mathcal{J}_n \stackrel{\text{def}}{=\!=} \sum_{\substack{r\geq 1, m\geq 0, k_1, k_2, \ldots, k_m \geq 2, \\ \sum_{j=1}^m \tilde\mu_{k_j} + \alpha_r \lambda_* = \tilde\mu_n}} \mathcal{F}_{r,m}(q_{k_1}, q_{k_2}, \ldots, q_{k_m}) \text{ in } \mathbb{R},$$

where $\mathcal{F}_{r,m}$ are m-linear mappings from $(\mathbb{R}^d)^m$ to $\mathbb{R}^d$.

3. Sketch of Proof

Proof (I). First asymptotic approximation

Proposition (Cao-H.-Kieu 2020)

Let $y(t)$ be a non-trivial, decaying solution. Then there exists a number $C_1 > 0$ such that

$$|y(t)| \leq C_1 e^{-\Lambda_1 t} \text{ for all } t \geq 0.$$

Moreover, for any $\varepsilon > 0$, there exists a number $C_2 = C_2(\varepsilon) > 0$ such that

$$|y(t)| \geq C_2 e^{-(\Lambda_d+\varepsilon)t} \text{ for all } t \geq 0.$$

Theorem (Cao-H.-Kieu 2020)

Let $y(t)$ be a non-trivial, decaying solution. Then there exist an eigenvalue λ_* of A and a corresponding eigenvector ξ_* such that

$$|y(t) - e^{-\lambda_* t}\xi_*| = \mathcal{O}(e^{-(\lambda_*+\delta)t}) \text{ for some } \delta > 0.$$

Foias-Saut use Dirichlet quotient $|A^{1/2}u|^2/|u|^2$. We have a new proof.

Proof (II). Infinite series expansion

Consider

$$F(x) \sim \sum_{k=1}^{\infty} F_k(x), \quad F_k \in \mathcal{H}_{\beta_k}(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d_0),$$

Definition

We define a set $\tilde{S} \subset [0,\infty)$ as follows. Let $\alpha_k = \beta_k - 1 > 0$ for $k \in \mathbb{N}$, and

$$\tilde{S} = \Big\{ \sum_{k=n_0}^{d_*} m_k(\lambda_k - \lambda_*) + \sum_{j=1}^{\infty} z_j \alpha_j \lambda_* : m_k, z_j \in \mathbb{Z}_+, \text{ with } z_j > 0 \text{ for only finitely many } j\text{'s} \Big\}.$$

The set $\tilde{S}$ has countably, infinitely many elements. Arrange $\tilde{S}$ as a sequence $(\tilde{\mu}_n)_{n=1}^{\infty}$ of non-negative and strictly increasing numbers. Set

$$\mu_n = \tilde{\mu}_n + \lambda_* \text{ for } n \in \mathbb{N}, \text{ and define } S = \{\mu_n : n \in \mathbb{N}\}.$$

Let $r \in \mathbb{N}$ and $s \in \mathbb{Z}_+$. Since F_r is a C^∞-function in a neighborhood of $\xi_* \neq 0$, we have the following Taylor's expansion, for any $h \in \mathbb{R}^d$,

$$F_r(\xi_* + h) = \sum_{m=0}^{s} \frac{1}{m!} D^m F_r(\xi_*) h^{(m)} + g_{r,s}(h),$$

where $D^m F_r(\xi_*)$ is the m-th order derivative of F_r at ξ_*, and

$$g_{r,s}(h) = \mathcal{O}(|h|^{s+1}) \text{ as } h \to 0.$$

For $m \geq 0$, denote

$$\mathcal{F}_{r,m} = \frac{1}{m!} D^m F_r(\xi_*).$$

When $m = 0$, $\mathcal{F}_{r,0} = F_r(\xi_*)$. When $m \geq 1$, $\mathcal{F}_{r,m}$ is an m-linear mapping from $(\mathbb{R}^d)^m$ to $\mathbb{R}^d$.

One has, for any $r, m \geq 1$, and $y_1, y_2, \ldots, y_m \in \mathbb{R}^d$, that

$$|\mathcal{F}_{r,m}(y_1, y_2, \ldots, y_m)| \leq \|\mathcal{F}_{r,m}\| \cdot |y_1| \cdot |y_2| \cdots |y_m|.$$

First Step ($N = 1$). By the first asymptotic approximation.
Induction Step. Let $y_n(t) = q_n(t)e^{-\mu_n t}$, $u_n(t) = y(t) - \sum_{k=1}^n y_k(t)$. By induction hypotheses,

$$u_N(t) = \mathcal{O}(e^{-(\mu_N+\delta_N)t}).$$

Let $w_N(t) = e^{\mu_{N+1}t}u_N(t)$. Then

$$w_N' + (A - \mu_{N+1}I_d)w_N = e^{\mu_{N+1}t}F(y) - e^{\mu_{N+1}t}\sum_{k=1}^{N}(Ay_k + y_k').$$

$$F(x) = \sum_{r=1}^{r_*} F_r(x) + \mathcal{O}(|x|^{\beta_{r_*}+\varepsilon_{r_*}}) \text{ as } x \to 0.$$

$$e^{\mu_{N+1}t}F(y(t)) = E(t) + e^{\mu_{N+1}t}\mathcal{O}(e^{-\lambda_*(\beta_{r_*}+\varepsilon_{r_*})t}),$$

where

$$E(t) = e^{\mu_{N+1}t}\sum_{r=1}^{r_*} F_r(y(t)).$$

Denote

$$\tilde{y}_k(t) = y_k(t)e^{\lambda_* t} = q_k(t)e^{-\tilde{\mu}_k t} \text{ and } \tilde{u}_k(t) = u_k(t)e^{\lambda_* t}.$$

Then

$$\tilde{u}_N(t) = u_N(t)e^{\lambda_* t} = \mathcal{O}(e^{-(\tilde{\mu}_N+\delta_N)t}), \quad \tilde{u}_1(t) = u_1(t)e^{\lambda_* t} = \mathcal{O}(e^{-\delta_1 t}).$$

Write

$$F_r(y(t)) = F_r(y_1 + u_1) = F_r\left(e^{-\lambda_* t}(\xi_* + \tilde{u}_1)\right) = e^{-\beta_r \lambda_* t} F_r(\xi_* + \tilde{u}_1).$$

By Taylor expansion about ξ_*:

$$F_r(y(t)) = e^{-\beta_r \lambda_* t}\left(F_r(\xi_*) + \sum_{m=1}^{s_*} \mathcal{F}_{r,m}\tilde{u}_1^{(m)}\right) + e^{-\beta_r \lambda_* t} g_{r,s_*}(\tilde{u}_1).$$

$$\mathcal{F}_{r,m}\tilde{u}_1^{(m)} = \mathcal{F}_{r,m}\Big(\sum_{k=2}^N \tilde{y}_k + \tilde{u}_N\Big)^{(m)}$$
$$= \mathcal{F}_{r,m}\Big(\sum_{k=2}^N \tilde{y}_k + \tilde{u}_N, \sum_{k=2}^N \tilde{y}_k + \tilde{u}_N, \ldots, \sum_{k=2}^N \tilde{y}_k + \tilde{u}_N\Big)$$
$$= \mathcal{F}_{r,m}\Big(\sum_{k=2}^N \tilde{y}_k\Big)^{(m)} + \sum_{\text{finitely many}} \mathcal{F}_{r,m}(z_1, \ldots, z_N).$$

$$\sum_{m=0}^{s_*} \mathcal{F}_{r,m}\tilde{u}_1^{(m)} = F_r(\xi_*) + \sum_{m=1}^{s_*} \mathcal{F}_{r,m}\Big(\sum_{k=2}^N \tilde{y}_k\Big)^{(m)} + \mathcal{O}(e^{-(\tilde{\mu}_N+\delta_N)t})$$
$$= \sum_{m=0}^{s_*} \sum_{k_1,\ldots,k_m \ge 2}^N \mathcal{F}_{r,m}(\tilde{y}_{k_1}, \tilde{y}_{k_2}, \ldots, \tilde{y}_{k_m}) + \mathcal{O}(e^{-(\tilde{\mu}_N+\delta_N)t})$$
$$= \sum_{m=0}^{s_*} \sum_{k_1,\ldots,k_m \ge 2}^N e^{-t\sum_{j=1}^m \tilde{\mu}_{k_j}} \mathcal{F}_{r,m}(q_{k_1}, q_{k_2}, \ldots, q_{k_m}) + \mathcal{O}(e^{-(\tilde{\mu}_N+\delta_N)t}).$$

Thus,

$$e^{-\beta_r\lambda_* t}\sum_{m=0}^{s_*}\mathcal{F}_{r,m}\tilde{u}_1^{(m)} = \sum_{m=0}^{s_*}\sum_{k_1,\ldots,k_m=2}^{N} e^{-t(\sum_{j=1}^m \tilde{\mu}_{k_j}+\beta_r\lambda_*)}\mathcal{F}_{r,m}(q_{k_1},q_{k_2},\ldots,q_{k_m}) + \mathcal{O}(e^{-(\tilde{\mu}_N+\beta_r\lambda_*+\delta_N)t}).$$

Exponent: $\mu=\tilde{\mu}_{k_1}+\ldots+\tilde{\mu}_{k_m}+\alpha_r\lambda_*\in\tilde{S}$, hence is some $\tilde{\mu}_p$. Then

$$\sum_{j=1}^m \tilde{\mu}_{k_j}+\beta_r\lambda_* = \mu+\lambda_* = \tilde{\mu}_p+\lambda_* = \mu_p \in S \quad \text{for some integer } p.$$

More manipulations:

$$w_N' + (A-\mu_{N+1}I_d)w_N = -e^{\mu_{N+1}t}\sum_{k=1}^N e^{-\mu_k t}\chi_k + \mathcal{J}_{N+1} + \mathcal{O}(e^{-\delta_N' t}),$$

where

$$\chi_1 = q_1' + (A-\mu_1 I_d)q_1, \quad \chi_k = q_k' + (A-\mu_k I_d)q_k - \mathcal{J}_k \text{ for } 2\le k\le N.$$

By the induction hypothesis, $\chi_k = 0$ for $2 \le k \le N$. Hence,

$$w_N' + (A - \mu_{N+1} I_d) w_N = \mathcal{J}_{N+1} + \mathcal{O}(e^{-\delta_N' t}),$$

where $\mathcal{J}_{N+1}(t)$ is a polynomial in t.

Lemma (Approximation Lemma)

Let $p(t)$ be an $\mathbb{R}^d$-valued polynomial and $g : [T, \infty) \to \mathbb{R}^d$, $|g(t)| = \mathcal{O}(e^{-\alpha t})$ for some $\alpha > 0$. Suppose $\lambda > 0$ and $y \in C([T, \infty), \mathbb{R}^d)$ is a solution of

$$y'(t) = -(A - \lambda I_d) y(t) + p(t) + g(t), \quad \text{for } t \in (T, \infty).$$

If $\lambda > \lambda_1$, assume further that

$$\lim_{t \to \infty} (e^{(\bar{\lambda} - \lambda)t} |y(t)|) = 0, \ \text{where } \bar{\lambda} = \max\{\lambda_j : 1 \le j \le d_*, \lambda_j < \lambda\}.$$

Then there exists a unique $\mathbb{R}^d$-valued polynomial $q(t)$ such that

$$q'(t) = -(A - \lambda I_d) q(t) + p(t) \ \text{for } t \in \mathbb{R}, \quad |y(t) - q(t)| = \mathcal{O}(e^{-\varepsilon t}).$$

By the Approximation Lemma, there exists polynomial $q_{N+1} : \mathbb{R} \to \mathbb{R}^d$ and a number $\delta_{N+1} > 0$ such that

$$|w_N(t) - q_{N+1}(t)| = \mathcal{O}(e^{-\delta_{N+1} t}).$$

Moreover $q_{N+1}(t)$ solves

$$q'_{N+1} + (A - \mu_{N+1} I_d) q_{N+1} = \mathcal{J}_{N+1},$$

that is, equation (4) holds for $n = N + 1$.
Multiplying estimate of $w_N - q_{N+1}$ by $e^{-\mu_{N+1} t}$ gives

$$\left| y(t) - \sum_{n=1}^{N+1} q_n(t) e^{-\mu_n t} \right| = \mathcal{O}(e^{-(\mu_{N+1} + \delta_{N+1}) t}),$$

which proves the statement for $N := N + 1$.

4. Extended results and examples

Extended results and examples

For $n \in \mathbb{N}$, $p \in [1, \infty)$ and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, the ℓ^p-norm of x is

$$\|x\|_p = \Big(\sum_{j=1}^{n} |x_j|^p \Big)^{1/p}.$$

Example

Let α be any number in $(0, \infty)$ that is not an even integer, and

$$F(x) = |x|^\alpha x \text{ for } x \in \mathbb{R}^d.$$

Example

Given a constant $d \times d$ matrix M_0, even numbers $p_1, p_2 \geq 2$, and real numbers $\alpha, \beta > 0$, let

$$F(x) = \frac{\|x\|_{p_1}^\alpha M_0 x}{1 + \|x\|_{p_2}^\beta} \text{ for } x \in \mathbb{R}^d.$$

For $x \in \mathbb{R}^d$ with $\|x\|_{p_2} < 1$, we expand $1/(1+\|x\|_{p_2}^{\beta})$, using the geometric series, and can verify that

$$F(x) \sim \sum_{k=1}^{\infty} (-1)^{k-1} \|x\|_{p_1}^{\alpha} \|x\|_{p_2}^{(k-1)\beta} M_0 x.$$

When $\|\cdot\|_{p_1} = \|\cdot\|_{p_2} = |\cdot|$, function F has form

$$F(x) \sim \sum_{k=1}^{\infty} c_k |x|^{\alpha+(k-1)\beta} M_0 x.$$

Others:

$$F(x) \sim \sum_{k=1}^{\infty} \|x\|_{p_k}^{\alpha_k} M_k x,$$

$$\Big| F(x) - \sum_{k=1}^{N_*} \|x\|_{p_k}^{\alpha_k} M_k x \Big| = \mathcal{O}(|x|^{\alpha_{N_*}+1+\bar{\varepsilon}}) \text{ as } x \to 0.$$

The last example, in general, only yields finite expansion for $y(t)$.

Example

Consider $d = 2$ and let

$$F(x_1, x_2) = (|x_1^3 - x_2^3|^{p_1} + |x_1^3 + x_2^3|^{p_1})^{\alpha/p_1} \cdot (|x_1 x_2|^{p_2} + |3x_1^2 - 2x_2^2|^{p_2})^{\beta/p_2} M_0(x_1, x_2)$$

where $p_1, p_2 \geq 2$ are even numbers, M_0 is a $\mathbb{R}^2$-valued homogeneous polynomials of degree $m_0 \in \mathbb{Z}_+$, and $\alpha, \beta > 0$.

Example

Consider the following system of ODEs in $\mathbb{R}^2$:

$$y_1' + 2y_1 + y_2 = |y|^{2/3} |y_1|^{1/2} y_2^3,$$
$$y_2' + y_1 + 2y_2 = \|y\|_{5/2}^{1/3} y_1 |y_2|^{1/4} \mathrm{sign}(y_2).$$

The corresponding matrix A has eigenvalues and bases of the corresponding eigenspaces as follows: $\lambda_1 = 1$, basis $\{(-1, 1)\}$, and $\lambda_2 = 3$, basis $\{(1, 1)\}$. Then any eigenvector of A belongs to $V = (\mathbb{R}_*)^2$. The corresponding function F belongs to $C^\infty(V)$.

THANK YOU!