Asymptotic expansions in time for solutions of Navier-Stokes equations of rotating fluids

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1. The Navier-Stokes systems

- NSE for rotating fluids
- Foias-Saut asymptotic expansion
- Poincaré wave and change of variables
- Forms of expansions

The Navier-Stokes equations

- The Navier-Stokes equations (NSE) in $\mathbb{R}^3$:

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu\Delta u + \nabla p + \Omega e_3 \times u = 0, \\ \mathrm{div}\ u = 0, \\ u(x,0) = u^0(x), \end{cases}$$

with viscosity $\nu > 0$, velocity field $u(x,t) \in \mathbb{R}^3$, pressure $p(x,t) \in \mathbb{R}$, initial velocity $u^0(x)$.

- Let $L > 0$ and $\Omega = (0,L)^3$. The L-periodic solutions:

$$u(x + Le_j) = u(x) \text{ for all } x \in \mathbb{R}^3, j = 1,2,3,$$

where $\{e_1, e_2, e_3\}$ is the canonical basis in $\mathbb{R}^3$.
Zero average condition

$$\int_\Omega u(x)dx = 0,$$

Throughout $L = 2\pi$ and $\nu = 1$.

Functional setting

Let $\mathcal{V}$ be the set of $\mathbb{R}^3$-valued 2π-periodic trigonometric polynomials which are divergence-free and satisfy the zero average condition.

$$H = \text{ closure of } \mathcal{V} \text{ in } L^2(\Omega)^3 = H^0(\Omega)^3,$$
$$V = \text{ closure of } \mathcal{V} \text{ in } H^1(\Omega)^3, \quad \mathcal{D}(A) = \text{ closure of } \mathcal{V} \text{ in } H^2(\Omega)^3.$$

Norm on H: $|u| = \|u\|_{L^2(\Omega)}$. Norm on V: $\|u\| = |\nabla u|$.
The Stokes operator:

$$Au = -\Delta u \text{ for all } u \in \mathcal{D}(A).$$

The bilinear mapping:

$$B(u,v) = \mathbb{P}_L(u \cdot \nabla v) \text{ for all } u, v \in \mathcal{D}(A).$$

$\mathbb{P}_L$ is the Leray projection from $L^2(\Omega)$ onto H.
Let $Ju = e_3 \times u$. The functional form of the NSE:

$$\frac{du(t)}{dt} + Au(t) + B(u(t),u(t)) + \Omega \mathbb{P}_L J \mathbb{P}_L u = 0, \ t > 0,$$

$$u(0) = u^0.$$

Non-rotation case $\Omega = 0$. Foias-Saut asymptotic expansion

Foias-Saut (1987) for a solution $u(t)$:

$$u(t) \sim \sum_{n=1}^{\infty} q_j(t) e^{-jt},$$

where $q_j(t)$ is a $\mathcal{V}$-valued polynomial in t. This means that for any $N \in \mathbb{N}$, $m \in \mathbb{N}$, the remainder $v_N(t) = u(t) - \sum_{j=1}^{N} q_j(t) e^{-jt}$ satisfies

$$\|v_N(t)\|_{H^m(\Omega)} = O(e^{-(N+\varepsilon)t})$$

as $t \to \infty$, for some $\varepsilon = \varepsilon_{N,m} > 0$.

Theorem (H.-Martinez 2017)

The Foias-Saut expansion holds in all Gevrey spaces:

$$\|e^{\sigma A^{1/2}} v_N(t)\|_{H^m(\Omega)} = O(e^{-(N+\varepsilon)t}),$$

for any $\sigma > 0$, $\varepsilon \in (0, 1)$.

Gevrey classes

- Spectrum of A is $\{|k|^2 : k \in \mathbb{Z}^3, k \neq 0\} = \{\Lambda_n\} \subset \mathbb{N}$.
- Additive semigroup is $\{\mu_n = n \in \mathbb{N}\}$.
- For $\alpha \geq 0$, $\sigma \geq 0$, define

$$A^\alpha e^{\sigma A^{1/2}} u = \sum_{\mathbf{k} \neq 0} |\mathbf{k}|^{2\alpha} \hat{u}(\mathbf{k}) e^{\sigma |\mathbf{k}|} e^{i\mathbf{k}\cdot\mathbf{x}}, \text{ for } u = \sum_{\mathbf{k} \neq 0} \hat{u}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \in H.$$

The domain of $A^\alpha e^{\sigma A^{1/2}}$ is

$$G_{\alpha,\sigma} = \mathcal{D}(A^\alpha e^{\sigma A^{1/2}}) = \{u \in H : |u|_{\alpha,\sigma} \stackrel{\text{def}}{=\!=} |A^\alpha e^{\sigma A^{1/2}} u| < \infty\}.$$

- Compare the Sobolev and Gevrey norms:

$$|A^\alpha u| = |(A^\alpha e^{-\sigma A^{1/2}}) e^{\sigma A^{1/2}} u| \leq \left(\frac{2\alpha}{e\sigma}\right)^{2\alpha} |e^{\sigma A^{1/2}} u|.$$

- For any numbers $\alpha \geq 1/2$, $\sigma \geq 0$, any functions $v, w \in G_{\alpha+1/2,\sigma}$ one has

$$|B(v,w)|_{\alpha,\sigma} \leq K^\alpha |v|_{\alpha+1/2,\sigma} |w|_{\alpha+1/2,\sigma}.$$

Poincaré wave

Let $S = \mathbb{P}_L J \mathbb{P}_L$. For $u^0 \in H$, set $w(t) = e^{\Omega t S} u^0$. Then $w(t) \in H$ solves

$$\frac{dw}{dt} = \Omega S w, \quad w(0) = u^0.$$

Note that S is anti-Hermitian and unitary on H, isometric on $D(A^\alpha)$ for all α.

Fourier-series:

$$e^{tS} u = \sum E_{\mathbf{k}}(t) \mathbf{u}_{\mathbf{k}},$$

where $E_{\mathbf{k}}(t)$ is a 3×3 matrix defined by

$$E_{\mathbf{k}}(t)\mathbf{z} = \cos(\tilde{k}_3 t)\,\mathbf{z} + \sin(\tilde{k}_3 t)\,\tilde{\mathbf{k}} \times \mathbf{z} \quad \forall \mathbf{z} \in \mathbb{C}^3,$$

with $\tilde{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$. We have

$$|E_{\mathbf{k}}(t)\mathbf{z}| = |\mathbf{z}|, \quad (E_{\mathbf{k}}(t))^* = E_{\mathbf{k}}(-t).$$

The semigroup e^{tS} is analytic in $t \in \mathbb{R}$, its adjoint operator is

$$(e^{tS})^* = e^{-tS},$$

unitary on H and isometric on $D(A^\alpha e^{\sigma A^{1/2}})$ for all α, σ.
Consequences,

$$|e^{tS} u|_{\alpha,\sigma} = |u|_{\alpha,\sigma},$$
$$(e^{tS})^* = e^{-tS}.$$

Sometimes, for convenient, we write

$$E_{\mathbf{k}}(t) = \cos(\tilde{k}_3 t)\mathbf{I}_3 - \frac{i}{|\mathbf{k}|}\sin(\tilde{k}_3 t)\mathbf{C}_{\mathbf{k}},$$

where $\mathbf{C}_{\mathbf{k}}$ is the matrix for the curl operator:

$$\mathbf{C}_{\mathbf{k}}\mathbf{z} = i\mathbf{k} \times \mathbf{z}.$$

Change of variables

Define for $t \in \mathbb{R}$,

$$B(t, u, v) = e^{tS} B(e^{-tS} u, e^{-tS} v),$$

$$B_\Omega(t, u, v) = B(\Omega t, u, v).$$

Let $u(t)$ be a solution of Rot-NSE. Set

$$v(t) = e^{t\Omega S} u(t).$$

Then $v(t)$ solves

$$\frac{dv}{dt} + Av + B_\Omega(t, v, v) = 0, \quad v(0) = v^0 = u^0.$$

Define

$$b(t, u, v, w) = b(e^{-tS} u, e^{-tS} v, e^{-tS} w), \quad b_\Omega(t, u, v, w) = b(\Omega t, u, v, w).$$

Note that

$$u(t) = e^{-t\Omega S} v(t).$$

We still have for all $t \in \mathbb{R}$ that

$$\langle B(t, u, v), w \rangle = \langle B(e^{-tS} u, e^{-tS} v), e^{-tS} w \rangle = -\langle B(e^{-tS} u, e^{-tS} w), e^{-tS} v \rangle,$$

thus,

$$\langle B(t, u, v), w \rangle = -\langle B(t, u, w), v \rangle.$$

Consequently,

$$\langle B(t, u, v), v \rangle = 0.$$

Lemma

For any numbers $\alpha \geq 1/2$, $\sigma \geq 0$, any functions $v, w \in G_{\alpha+1/2,\sigma}$ and any $t \in \mathbb{R}$, one has

$$|B(t, v, w)|_{\alpha,\sigma} \leq K^{\alpha} |v|_{\alpha+1/2,\sigma} |w|_{\alpha+1/2,\sigma}.$$

Leray-Hopf weak solutions

A *Leray-Hopf weak solution* $v(t)$ of Wav-NSE is a mapping from $[0,\infty)$ to H such that

$$v \in C([0,\infty), H_{\rm w}) \cap L^2_{\rm loc}([0,\infty), V), \quad v' \in L^{4/3}_{\rm loc}([0,\infty), V'),$$

and satisfies

$$\frac{d}{dt}\langle v(t), w\rangle + \langle\langle v(t), w\rangle\rangle + b_\Omega(t, v(t), v(t), w) = 0$$

in the distribution sense in $(0,\infty)$, for all $w \in V$, and the energy inequality

$$\frac{1}{2}|v(t)|^2 + \int_{t_0}^t \|v(\tau)\|^2 d\tau \le \frac{1}{2}|v(t_0)|^2$$

holds for $t_0 = 0$ and almost all $t_0 \in (0,\infty)$, and all $t \ge t_0$.

S- and SS- polynomials

Let X be a linear space.

1. A function $g : \mathbb{R} \to X$ is an X-valued S-polynomial if it is a finite sum of the functions in the collection
$$\Big\{t^m(\cos(\omega t))Z,\ t^m(\sin(\omega t))Z : m \in \mathbb{N} \cup \{0\},\ \omega \in \mathbb{R},\ Z \in X\Big\}.$$
2. A function $g : \mathbb{R} \to X$ is an X-valued SS-polynomial if it is a finite sum of the functions
$$t^m f(t) g_1(t) g_2(t) g_3(t) Z,$$
where $m \in \mathbb{N} \cup \{0\}$, $Z \in X$, f is in
$$S_1 \stackrel{\text{def}}{=\!=} \{\cos(\omega t), \sin(\omega t) : \omega \in \mathbb{R}\},$$
and g_1, g_2, g_3 are in S_1 or
$$\{\cos(a\cos(bt)), \cos(a\sin(bt)), \sin(a\cos(bt)), \sin(a\sin(bt)) : a, b \in \mathbb{R}\}.$$

Note that if $g : \mathbb{R} \to X$ is an X-valued S-polynomial then we can write

$$g(t) = \sum_{j=1}^{N} \big(A_j \cos(\omega_j t) + B_j \sin(\omega_j t) \big) t^{m_j}$$

for some integer $N \geq 1$, coefficients A_j, B_j belonging to X, real numbers ω_j, and non-negative integers m_j with $m_j \leq m_{j+1}$ for $1 \leq j \leq N-1$, or in another form:

$$g(t) = \sum_{n=0}^{N} g_n(t) t^n, \text{ where } g_n(t) = \sum_{j=0}^{N_n} \big(A_{n,j} \cos(\omega_{n,j} t) + B_{n,j} \sin(\omega_{n,j} t) \big),$$

with non-negative numbers $\omega_{n,j}$'s are strictly increasing in $j \geq 0$.

Asymptotic expansions

Let $(X, \|\cdot\|)$ be a normed space and $(\alpha_n)_{n=1}^{\infty}$ be a sequence of strictly increasing non-negative numbers. A function $f : [T, \infty) \to X$, for some $T \in \mathbb{R}$, is said to have an asymptotic expansion

$$f(t) \sim \sum_{n=1}^{\infty} f_n(t) e^{-\alpha_n t} \quad \text{in } X,$$

where $f_n(t)$ is an X-valued polynomial, or S-polynomial, or SS-polynomial, if one has, for any $N \geq 1$, that

$$\left\| f(t) - \sum_{n=1}^{N} f_n(t) e^{-\alpha_n t} \right\| = \mathcal{O}(e^{-(\alpha_N + \varepsilon_N)t}) \quad \text{as } t \to \infty,$$

for some $\varepsilon_N > 0$.

2. Main results

- Expansions of the solutions with zero averages
- Expansions without the zero average condition

With zero average condition

Two main expansions, one for $v(t)$ and one for $u(t)$.

Theorem

For any Leray-Hopf weak solution $v(t)$ of Wav-NSE, there exist $\mathcal{V}$-valued S-polynomials q_n's, for all $n \in \mathbb{N}$, such that if $\alpha, \sigma > 0$ and $N \geq 1$ then

$$\left| v(t) - \sum_{n=1}^{N} q_n(t) e^{-\mu_n t} \right|_{\alpha,\sigma} = \mathcal{O}\left(e^{-\mu t}\right) \quad as \quad t \to \infty, \quad \forall \mu \in (\mu_N, \mu_{N+1}).$$

Since $u(t) = e^{-\Omega t S} v(t)$, we immediately obtain the expansion for $u(t)$.

Theorem

Let $u(t)$ be any Leray-Hopf weak solution of Rot-NSE. Then there exist $\mathcal{V}$-valued S-polynomials Q_n's, for all $n \in \mathbb{N}$, such that it holds, for any $\alpha, \sigma > 0$ and $N \geq 1$, that

$$\left| u(t) - \sum_{n=1}^{N} Q_n(t) e^{-\mu_n t} \right|_{\alpha,\sigma} = \mathcal{O}\left(e^{-\mu t}\right) \quad as \quad t \to \infty, \quad \forall \mu \in (\mu_N, \mu_{N+1}).$$

Proof. Let $v(t) = e^{\Omega t S} u(t)$. Then $v(t)$ is a Leray-Hopf weak solution of Wav-NSE. Hence $v(t)$ admits an asymptotic expansion. Rewrite the remainder estimate in terns of $u(t)$ as

$$\left| e^{\Omega t S} \left(u(t) - \sum_{n=1}^{N} q_n(t) e^{-\Omega t S} e^{-\mu_n t} \right) \right|_{\alpha,\sigma} = \left| u(t) - \sum_{n=1}^{N} Q_n(t) e^{-\mu_n t} \right|_{\alpha,\sigma},$$

where $Q_n(t) = e^{-\Omega t S} q_n(t)$ are also S-polynomials.

Without the zero average condition

Galilean transformation. For $t \geq 0$, let

$$U(t) = \frac{1}{L^3} \int u(x,t)dx.$$

When $t = 0$, denote

$$U_0 = U(0) = \frac{1}{L^3} \int u(x,0)dx = \frac{1}{L^3} \int u_0(x)dx.$$

Integrating the equation Rot-NSE over the domain gives

$$U'(t) + \Omega J U(t) = 0.$$

Hence,

$$U(t) = e^{-\Omega t J} U_0 = \begin{pmatrix} \cos(\Omega t) & \sin(\Omega t) & 0 \\ -\sin(\Omega t) & \cos(\Omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix} U_0.$$

Let

$$V(t) = \int_0^t U(\tau)d\tau = \frac{1}{\Omega}\begin{pmatrix} \sin(\Omega t) & 1-\cos(\Omega t) & 0 \\ \cos(\Omega t)-1 & \sin(\Omega t) & 0 \\ 0 & 0 & \Omega t \end{pmatrix} U_0.$$

Then $V(0)=0$ and $V'(t)=U(t)$. Define for $t\ge 0$,

$$w(x,t) = u(x+V(t),t) - U(t), \quad \vartheta(x,t) \mapsto p(x+V(t),t),$$

Then $w(\cdot,t)$ has zero average for each t, (w,ϑ) is a $\mathbf{L}$-periodic solution of the NSE:

$$w_t - \Delta w + (w\cdot\nabla)w + \Omega J w = -\nabla\vartheta,$$

$$\mathrm{div}\ w = 0,$$

$$w(x,0) = w_0(x) \stackrel{\mathrm{def}}{=\!=} u_0(x) - U_0(x).$$

Theorem

Let $u(x,t) \in C^{2,1}_{x,t}(\mathbb{R}^2 \times (0,\infty)) \cap C(\mathbb{R}^3 \times [0,\infty))$ be a L-periodic solution of the NSE. There exist $\mathcal{V}$-valued SS-polynomials $\tilde{Q}_n(t)$'s, for all $n \in \mathbb{N}$, such that

$$u(t) \sim U(t) + \sum_{n=1}^{\infty} \tilde{Q}(t) e^{-\mu_n t} \text{ in } \tilde{G}_{\alpha,\sigma} \text{ for all } \alpha, \sigma > 0.$$

Formal proof. We have

$$u(x,t) = U(t) + w(x - V(t), t) \sim U(t) + \sum_{n=1}^{\infty} Q_n(x - V(t), t) e^{-\mu_n t}.$$

Note

$$e^{ik\cdot(x - V(t))} = e^{ik\cdot x} e^{-ik\cdot V(t)} = e^{ik\cdot x}(\cos(k \cdot V(t)) - i \sin(k \cdot V(t))).$$

Suppose

$$Q_n(x,t) = \sum \hat{Q}_{n,k}(t) e^{ik\cdot x}.$$

Then

$$Q_n(x - V(t), t) = \sum \hat{Q}_{n,k}(t) e^{ik\cdot x} e^{ik\cdot x} e^{-ik\cdot V(t)}$$
$$= \sum \hat{Q}_{n,k}(t) e^{ik\cdot x}(\cos(k \cdot V(t)) - i \sin(k \cdot V(t))).$$

Because $V(t)$ already contains $\cos(bt)$ and $\sin(bt)$ terms, so $u(x, t)$ will contain terms of the forms

$$\cos(a\cos(bt)), \cos(a\sin(bt)), \sin(a\cos(bt)), \sin(a\sin(bt)),$$

and

$$t^m\cos(a\cos(bt)), t^m\cos(a\sin(bt)), t^m\sin(a\cos(bt)), t^m\sin(a\sin(bt)).$$

3. Proofs

Proof for the zero average case

Theorem (same as H.-Martinez 2017)

Let $v^0 \in H$ and $v(t)$ be a Leray-Hopf weak solution of Wav-NSE. For any $\sigma > 0$, there exist $T, D_\sigma > 0$ such that

$$|v(t)|_{1/2,\sigma+1} \le D_\sigma e^{-t} \quad \forall t \ge T.$$

Moreover, for any $\alpha \ge 0$ there exists $D_{\alpha,\sigma} > 0$ such that

$$|v(t)|_{\alpha+1/2,\sigma} \le D_{\alpha,\sigma} e^{-t} \quad \forall\ t \ge T.$$

Induction statement

Let $\sigma > 0$ be fixed. We prove the following statement

($\mathcal{T}_N$)

For any $N \geq 1$, there exist $\mathcal{V}$-valued S-polynomials q_n's for $n = 1, 2, \ldots, N$, such that

$$\Big| v(t) - \sum_{n=1}^{N} q_n(t) e^{-\mu_n t} \Big|_{\alpha,\sigma} = \mathcal{O}(e^{-(\mu_N+\varepsilon)t}) \text{ as } t \to \infty,$$

for all $\alpha > 0$, and some $\varepsilon = \varepsilon_{N,\alpha} > 0$. Moreover, each $v_n(t) \stackrel{\text{def}}{=\!=} q_n(t) e^{-\mu_n t}$, for $n = 1, 2, \ldots, N$, solves

$$v_n' + Av_n + \sum_{\substack{1 \leq m,k \leq n-1 \\ \mu_m + \mu_k = \mu_n}} B_\Omega(t, v_m, v_k) = 0 \quad \forall t \in \mathbb{R}.$$

First step $N = 1$

Let $w_0 = e^{\mu_1 t} v(t)$. We have

$$w_0' + (A - \mu_1) w_0 = H_0(t) \stackrel{\text{def}}{=\!=} -e^{\mu_1 t} B_\Omega(t, v(t), v(t)).$$

There exist $T_0, d_0 > 0$ such that

$$|v(t)|_{\alpha+1/2,\sigma} \le d_0 e^{-\mu_1 t} \quad \forall t \ge T_0.$$

Hence

$$|H_0(T_0 + t)|_{\alpha,\sigma} \le e^{\mu_1 (T_0+t)} K^\alpha |v(T_0 + t)|^2_{\alpha+1/2,\sigma} \le M_0 e^{-\mu_1 t} \quad \forall t \ge 0.$$

For $k \in \mathbb{N}$, taking the projection R_{Λ_k} gives

$$(R_{\Lambda_k} w_0)' + (\Lambda_k - \mu_1) R_{\Lambda_k} w_0 = R_{\Lambda_k} H_0(t).$$

Approximation lemma

Let $(X, \|\cdot\|)$ be a Banach space. Suppose $y(t)$ is a function in $C([0,\infty),X)$ that solves the following ODE

$$y'(t) + \beta y(t) = p(t) + g(t)$$

in the X-valued distribution sense on $(0,\infty)$. Here, $\beta \in \mathbb{R}$ is a fixed constant, $p(t)$ is an X-valued S-polynomial, and $g \in L^1_{\rm loc}([0,\infty),X)$ satisfies

$$\|g(t)\| \le Me^{-\delta t} \quad \forall t \ge 0, \quad \text{for some } M,\delta > 0.$$

Define $q(t)$, for $t \in \mathbb{R}$, by

$$q(t) = \begin{cases} e^{-\beta t}\int_{-\infty}^t e^{\beta\tau}p(\tau)d\tau & \text{if } \beta > 0,\\ y(0) + \int_0^\infty g(\tau)d\tau + \int_0^t p(\tau)d\tau & \text{if } \beta = 0,\\ -e^{-\beta t}\int_t^\infty e^{\beta\tau}p(\tau)d\tau & \text{if } \beta < 0. \end{cases}$$

Then $q(t)$ is an X-valued S-polynomial that satisfies

$$q'(t) + \beta q(t) = p(t), \quad t \in \mathbb{R},$$

and the following estimates hold:

Approximation lemma (continued)

1. If $\beta > 0$ then
$$\|y(t) - q(t)\|^2 \le 2e^{-2\beta t}\|y(0) - q(0)\|^2 + 2t\int_0^t e^{-2\beta(t-\tau)}\|g(\tau)\|^2 d\tau.$$

2. If either
   (a) $\beta = 0$, or
   (b) $\beta < 0$ and
$$\lim_{t\to\infty}(e^{\beta t}\|y(t)\|) = 0,$$
   then
$$\|y(t) - q(t)\|^2 \le \Big(\frac{M}{\delta - \beta}\Big)^2 e^{-2\delta t}.$$

$N = 1$ (cont.)

We apply the Lemma to space $X = R_{\Lambda_k} H$ with X-norm $\|\cdot\| = |\cdot|_{\alpha,\sigma}$, solution $y(t) = R_{\Lambda_k} w_0(T_0 + t)$, S-polynomial $p(t) \equiv 0$, constant $\beta = \Lambda_k - \mu_1 \geq 0$, function $g(t) = R_{\Lambda_k} H_0(T_0 + t)$ and numbers $M = M_0$, $\delta = \mu_1$ in (27).

When $k = 1$, we have $\beta = 0$, then by Lemma (ii), it follows that

$$|R_{\Lambda_1} w_0(T_0 + t) - \xi_1|_{\alpha,\sigma} = \mathcal{O}(e^{-\mu_1 t}),$$

$$\xi_1 = R_{\Lambda_1} w_0(T_0) + \int_0^\infty e^{\mu_1 \tau} R_{\Lambda_1} H_0(T_0 + \tau) d\tau,$$

which exists and belongs to $R_{\Lambda_1} H$.

When $k \geq 2$,

$$|(\mathrm{Id} - R_{\Lambda_1}) w_0(T + t)|^2_{\alpha,\sigma} \leq 2e^{-2(\mu_2 - \mu_1)t} \big(|w_0(T)|^2_{\alpha,\sigma} + 2Mt^2 \big).$$

Thus,

$$|w_0(t) - \xi_1|_{\alpha,\sigma} \leq |R_{\Lambda_1} w_0(t) - \xi_1|_{\alpha,\sigma} + |(\mathrm{Id} - R_{\Lambda_1}) w_0(t)|_{\alpha,\sigma} = \mathcal{O}(e^{-\varepsilon t}).$$

We obtain $q_1(t) \equiv \xi_1$.

Induction step

Let $v_n(t) = q_n(t)e^{-\mu_n t}$, $\bar{v}_N(t) = \sum_{n=1}^N v_n(t)$ and $\tilde{v}_N(t) = v(t) - \bar{v}_N(t)$.
Let $w_N(t) = e^{\mu_{N+1} t}\tilde{v}_N(t)$, and $w_{N,k}(t) = R_{\Lambda_k} w_N(t)$ for $k \in \mathbb{N}$.
We have

$$\frac{d}{dt} w_{N,k} + (\Lambda_k - \mu_{N+1}) w_{N,k} = - \sum_{\mu_m+\mu_j=\mu_{N+1}} R_{\Lambda_k} B_\Omega(t, q_m, q_j) + R_{\Lambda_k} H_N(t).$$

There exist $T_N > 0$ and $M_N > 0$ such that

$$|H_N(T_N + t)|_{\alpha,\sigma} \le M_N e^{-\delta_N t} \quad \forall t \ge 0.$$

We will apply the Lemma again to space $X = R_{\Lambda_k} H$ with X-norm $\| \cdot \| = | \cdot |_{\alpha,\sigma}$, solution $y(t) = w_{N,k}(T_N + t)$, constant $\beta = \Lambda_k - \mu_{N+1}$, S-polynomial

$$p(t) = - \sum_{\mu_m + \mu_j = \mu_{N+1}} R_{\Lambda_k} B_\Omega(T_N + t, q_m(T_N + t), q_j(T_N + t)),$$

function $g(t) = R_{\Lambda_k} H_N(T_N + t)$, numbers $M = M_N$ and $\delta = \delta_N$.
We obtain S-polynomials $p_{N+1,k}(t)$ to approximate $w_{N,k}(T_N + t)$.
Define

$$q_{N+1}(t) = \sum_{k=1}^{\infty} p_{N+1,k}(t - T).$$

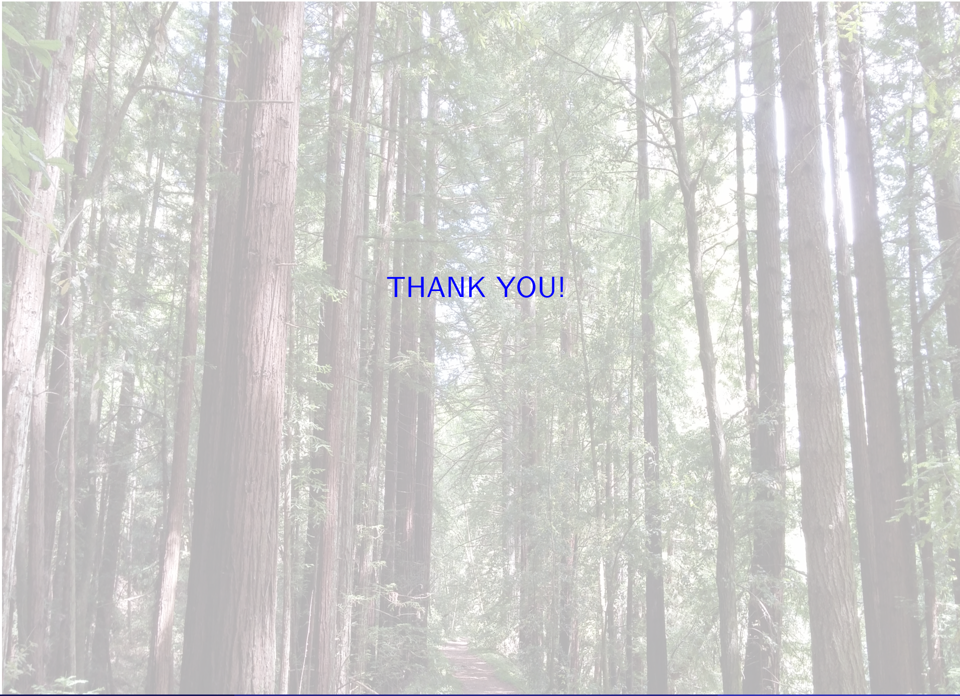
Then $R_{\Lambda_k} p_{N+1}(t) = p_{N+1,k}(t - T)$.
We have remainder estimate

$$|w_N(t + T) - q_{N+1}(t + T)|^2_{\alpha,\sigma} = \mathcal{O}(e^{-\delta_N t}).$$

which implies

$$|w_N(t) - q_{N+1}(t)|_{\alpha,\sigma} = \mathcal{O}(e^{-\delta_N t/2}).$$

Need to check the ODE for $q_{N+1}(t)$, but it is OK.



THANK YOU!