Asymptotic expansions in Gevrey spaces for solutions of Navier-Stokes equations in periodic domains

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1. Introduction

- Navier-Stokes equations
- Foias-Saut asymptotic expansion
Navier-Stokes equations (NSE) in $\mathbb{R}^3$ with a potential body force

\[
\begin{aligned}
\frac{\partial u}{\partial t} + (u \cdot \nabla) u - \nu \Delta u + \nabla p &= f(x, t), \\
\text{div } u &= 0, \\
\ u(x, 0) &= u^0(x),
\end{aligned}
\]

$\nu > 0$ is the kinematic viscosity, 
$u = (u_1, u_2, u_3)$ is the unknown velocity field, 
$p \in \mathbb{R}$ is the unknown pressure, 
$f(x, t)$ is the body force, 
$u^0$ is the initial velocity.
Let $L > 0$ and $\Omega = (0, L)^3$. The $L$-periodic solutions:

$$u(x + Le^j) = u(x) \text{ for all } x \in \mathbb{R}^3, j = 1, 2, 3,$$

where $\{e_1, e_2, e_3\}$ is the canonical basis in $\mathbb{R}^3$.

Zero average condition

$$\int_{\Omega} u(x) dx = 0,$$

Throughout $L = 2\pi$ and $\nu = 1$. 
Let $\mathcal{V}$ be the set of $\mathbb{R}^3$-valued $2\pi$-periodic trigonometric polynomials which are divergence-free and satisfy the zero average condition.

$$H = \text{closure of } \mathcal{V} \text{ in } L^2(\Omega)^3 = H^0(\Omega)^3,$$

$$V = \text{closure of } \mathcal{V} \text{ in } H^1(\Omega)^3, \quad \mathcal{D}(A) = \text{closure of } \mathcal{V} \text{ in } H^2(\Omega)^3.$$

Norm on $H$: $|u| = \|u\|_{L^2(\Omega)}$. Norm on $V$: $\|u\| = |\nabla u|$. The Stokes operator:

$$Au = -\Delta u \text{ for all } u \in \mathcal{D}(A).$$

The bilinear mapping:

$$B(u, \nu) = \mathbb{P}_L (u \cdot \nabla \nu) \text{ for all } u, \nu \in \mathcal{D}(A).$$

$\mathbb{P}_L$ is the Leray projection from $L^2(\Omega)$ onto $H$. Spectrum of $A$:

$$\sigma(A) = \{|k|^2, 0 \neq k \in \mathbb{Z}^3\}.$$ 

Denote by $R_N H$ the eigenspace of $A$ corresponding to $N$. 

WLOG, assume $f(t) = \mathbb{P}_L f(t)$.
The functional form of the NSE:

$$\frac{du(t)}{dt} + Au(t) + B(u(t), u(t)) = f(t), \quad t > 0,$$

$$u(0) = u^0.$$
Case $f = 0$.

- Foias-Saut (1987) for a solution $u(t)$:
  \[ u(t) \sim \sum_{n=1}^{\infty} q_j(t) e^{-jt}, \]
  where $q_j(t)$ is a $\mathcal{V}$-valued polynomial in $t$. This means that for any $N \in \mathbb{N}$, $m \in \mathbb{N}$, the remainder $v_N(t) = u(t) - \sum_{j=1}^{N} q_j(t) e^{-jt}$ satisfies
  \[ \|v_N(t)\|_{H^m(\Omega)} = O(e^{-(N+\varepsilon)t}) \]
  as $t \to \infty$, for some $\varepsilon = \varepsilon_{N,m} > 0$.

- H.-Martinez (2017) proved that the expansion holds in Gevrey spaces:
  \[ \|e^{\sigma A^{1/2}} v_N(t)\|_{H^m(\Omega)} = O(e^{-(N+\varepsilon)t}), \]
  for any $\sigma > 0$, $\varepsilon \in (0, 1)$.
  They used Gevrey norm techniques (Foias-Temam 1989) to simplify the proof.
2. Main results
Gevrey classes

For $\alpha \geq 0$, $\sigma \geq 0$, define

$$A^\alpha e^{\sigma A^{1/2}} u = \sum_{k \neq 0} |k|^{2\alpha} \hat{u}(k) e^{\sigma |k|} e^{i k \cdot x}, \text{ for } u = \sum_{k \neq 0} \hat{u}(k) e^{i k \cdot x} \in H.$$ 

The domain of $A^\alpha e^{\sigma A^{1/2}}$ is

$$G_{\alpha, \sigma} = \mathcal{D}(A^\alpha e^{\sigma A^{1/2}}) = \{ u \in H : |u|_{\alpha, \sigma} \overset{\text{def}}{=} |A^\alpha e^{\sigma A^{1/2}} u| < \infty \}.$$ 

- Compare the Sobolev and Gevrey norms:

$$|A^\alpha u| = |(A^\alpha e^{-\sigma A^{1/2}}) e^{\sigma A^{1/2}} u| \leq \left( \frac{2\alpha}{e^\sigma} \right)^{2\alpha} |e^{\sigma A^{1/2}} u|.$$
Denote for $\sigma \in \mathbb{R}$ the space

$$E^{\infty,\sigma} = \bigcap_{\alpha \geq 0} G_{\alpha,\sigma} = \bigcap_{m \in \mathbb{N}} G_{m,\sigma}.$$  

We will say that an asymptotic expansion holds in $E^{\infty,\sigma}$ if it holds in $G_{\alpha,\sigma}$ for all $\alpha \geq 0$.

Denote by $P^{\alpha,\sigma}$ the space of $G_{\alpha,\sigma}$-valued polynomials in case $\alpha \in \mathbb{R}$, and the space of $E^{\infty,\sigma}$-valued polynomials in case $\alpha = \infty$. 

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Let \( X \) be a real vector space.

(a) An \( X \)-valued polynomial is a function \( t \in \mathbb{R} \mapsto \sum_{n=1}^{d} a_n t^n \), for some \( d \geq 0 \), and \( a_n \)'s belonging to \( X \).

(b) In case \( \| \cdot \| \) is a norm on \( X \), a function \( g(t) \) from \( (0, \infty) \) to \( X \) is said to have the asymptotic expansion

\[
g(t) \sim \sum_{n=1}^{\infty} g_n(t) e^{-nt} \text{ in } X,
\]

where \( g_n(t) \)'s are \( X \)-valued polynomials, if for all \( N \geq 1 \), there exists \( \varepsilon_N > 0 \) such that

\[
\left\| g(t) - \sum_{n=1}^{N} g_n(t) e^{-nt} \right\| = \mathcal{O}(e^{-(N+\varepsilon_N)t}) \text{ as } t \to \infty.
\]
Exponentially decaying forces

**Assumptions.**

(A1) The function $f(t)$ is continuous from $[0, \infty)$ to $H$.

(A2) There are a number $\sigma_0 \geq 0$, $E^{\infty, \sigma_0}$-valued polynomials $f_n(t)$ for all $n \geq 1$, and a sequence of numbers $\delta_n \in (0, 1)$ for all $n \geq 1$ such that for each $N \geq 1$

$$\left| f(t) - \sum_{n=1}^{N} f_n(t) e^{-nt} \right|_{\alpha, \sigma_0} = O(e^{-(N+\delta_N)t}) \quad \text{as } t \to \infty, \quad \text{for all } \alpha \geq 0.$$ 

That is, the force $f(t)$ admits the following expansion in $G_{\alpha, \sigma_0}$ for all $\alpha \geq 0$:

$$f(t) \sim \sum_{n=1}^{\infty} f_n(t)e^{-nt}.$$
The followings are direct consequences of the Assumptions.

(a) For each $\alpha > 0$ that $f(t)$ belongs to $G_{\alpha,\sigma_0}$ for $t$ large.

(b) When $N = 1$,

$$|f(t) - f_1(t)e^{-t}|_{\alpha,\sigma_0} = O(e^{-(1+\delta_1)t}).$$

Since $f_1(t)$ is a polynomial, it follows that

$$|f(t)|_{\alpha,\sigma_0} = O(e^{-\lambda t}) \quad \forall \lambda \in (0, 1), \forall \alpha > 0.$$

(c) Combining with Assumption (A1), for each $\lambda \in (0, 1)$, there is $M_\lambda > 0$ such that

$$|f(t)| \leq M_\lambda e^{-\lambda t} \quad \forall t \geq 0.$$
Theorem (Asymptotic expansion, H.-Martinez 2018)

Let $u(t)$ be a Leray-Hopf weak solution. Then there exist polynomials $q_n \in P^{\infty,\sigma_0}$, for all $n \geq 1$, such that $u(t)$ has the asymptotic expansion

$$u(t) \sim \sum_{n=1}^{\infty} q_n(t) e^{-nt} \quad \text{in } E^{\infty,\sigma_0}.$$

Moreover, the mappings

$$u_n(t) \overset{\text{def}}{=} q_n(t) e^{-nt} \quad \text{and} \quad F_n(t) \overset{\text{def}}{=} f_n(t) e^{-nt},$$

satisfy the following ordinary differential equations in the space $E^{\infty,\sigma_0}$

$$\frac{d}{dt} u_n(t) + Au_n(t) + \sum_{k,m \geq 1 \atop k+m=n} B(u_k(t), u_m(t)) = F_n(t), \quad t \in \mathbb{R}, \quad (\star)$$

for all $n \geq 1$. 
Finite asymptotic approximation

**Theorem (Finite asymptotic approximation, H.-Martinez 2018)**

Suppose there exist an integer \( N_\ast \geq 1 \), real numbers \( \sigma_0 \geq 0 \), \( \mu_\ast \geq \alpha_\ast \geq N_\ast / 2 \), and, for any \( 1 \leq n \leq N_\ast \), numbers \( \delta_n \in (0, 1) \) and polynomials \( f_n \in P^{\mu_n, \sigma_0} \), such that

\[
\left| f(t) - \sum_{n=1}^{N} f_n(t)e^{-nt} \right|_{\alpha_N, \sigma_0} = O(e^{-(N+\delta_N)t}) \quad \text{as } t \to \infty,
\]

for \( 1 \leq N \leq N_\ast \), where

\[
\mu_n = \mu_\ast - (n - 1)/2, \quad \alpha_n = \alpha_\ast - (n - 1)/2.
\]
Theorem (continued)

Let $u(t)$ be a Leray-Hopf weak.

(i) Then there exist polynomials $q_n \in P^{\mu_n+1,\sigma_0}$, for $1 \leq n \leq N_*$, such that one has for $1 \leq N \leq N_*$ that

$$\left| u(t) - \sum_{n=1}^{N} q_n(t)e^{-nt} \right|_{\alpha_N,\sigma_0} = O(e^{-(N+\varepsilon)t}) \quad \text{as } t \to \infty, \quad \forall \varepsilon \in (0, \delta_N^*),$$

where $\delta_N^* = \min\{\delta_1, \delta_2, \ldots, \delta_N\}$.

Moreover, the ODEs

$$\frac{d}{dt} u_n(t) + A u_n(t) + \sum_{k,m \geq 1 \atop k+m=n} B(u_k(t), u_m(t)) = F_n(t), \quad t \in \mathbb{R}, \quad (\star)$$

hold in the corresponding space $G^{\mu_n,\sigma_0}$ for $1 \leq n \leq N_*$.

(ii) In particular, if all $f_n(t)$'s belong to $\mathcal{V}$, resp., $E^{\infty,\sigma_0}$, then so do all $q_n(t)$'s, and the ODEs $(\star)$ hold in $\mathcal{V}$, resp., $E^{\infty,\sigma_0}$.
3. Sketch of proofs

The resolution of quich equations out of $D'Wells$'s
in his indicacn, before employing anywiore.

Suppose $x = 8a + 8e$. $y = x^3 = 8a^3 + 8ae + 8e^3$.

Or $x^3 = 0 + 3a + 8a^3 + 8ae + 8e^3$. That is making $a^3 + e^3 = \Phi$.

Again suppose $y = x^6$. Then

Again suppose $x = a + e$. $y^2 = x^3 = a^3 + 3a^2e + 3ae^2 + e^3$.

But in the first of these $f = 3a^2 + 3ae + e^2 = 0$.

Or $e = e^2 = a - 3e$. Therefore $e^6 = 9e^3 - \frac{p^3}{27}$.

But here observe that Dr. Walls would Bcucue
$qf$ since in the first of these two cases formal algebra
(viz. when $y^2$ equation had $3$ roots) $y^2$ rule failed
as if it were impossible for $y^2$ equation to have roots
when $y$ had $3$ root, therefore $y^2$ fault is in Algebra.

Therefore when $y^2$ Analysis leads us to an absolu-
tely impossible we ought not to concluce $y^2$ being imposs-
ible, until we have tried all $y^2$ ways. $y^2$ may brc.

But let us answer $y^2$ fault is root in $y^2$ Analysis.

But in this example, but his operatlon. for $D$ when $q^2 = e$
in equation $x^3 + px + q = 0$, there is root, the suppposted to be
have but one root, viz. $x = 8a + 8e$. But since $q^2$ equation

Be generated according to a $y^2$ suppose it.
Estimates for the bilinear form

Lemma

If $\alpha \geq 1/2$ and $\sigma \geq 0$ then

$$|B(u, v)|_{\alpha, \sigma} \leq K^\alpha |u|_{\alpha+1/2, \sigma} |v|_{\alpha+1/2, \sigma},$$

for all $u, v \in G_{\alpha+1/2, \sigma}$.

Proof. Let $u, v, w$ be $H$ with

$$u = \sum_{k \neq 0} \hat{u}(k)e^{-ik \cdot x}, \quad v = \sum_{k \neq 0} \hat{v}(k)e^{-ik \cdot x}, \quad w = \sum_{k \neq 0} \hat{w}(k)e^{-ik \cdot x}.$$

Define the scalar functions

$$u_* = \sum_{k \neq 0} |\hat{u}(k)|e^{-ik \cdot x}, \quad v_* = \sum_{k \neq 0} |\hat{v}(k)|e^{-ik \cdot x}, \quad w_* = \sum_{k \neq 0} |\hat{w}(k)|e^{-ik \cdot x}.$$

Then

$$|A^\alpha e^{\sigma A^{1/2}} u| = \left|( -\Delta \right)^{\alpha} e^{\sigma ( -\Delta)^{1/2}} u_*|$$

for all $\alpha, \sigma \geq 0$. 

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We have
\[ I = \langle A^{\alpha} e^{\sigma A^{1/2}} B(u, v), w \rangle = 8\pi^3 \sum_{k+l+m=0} |m|^{2\alpha} e^{\sigma|m|} (\hat{u}(k) \cdot l) (\hat{v}(l) \cdot \hat{w}(m)). \]

Since
\[ |m|^{2\alpha} = |k + l|^{2\alpha} \leq 2^{2\alpha} (|k|^{2\alpha} + |l|^{2\alpha}) \quad \text{and} \quad e^{\sigma|m|} \leq e^{\sigma|k|} e^{\sigma|l|}, \]

it follows that
\[ I \leq 8\pi^3 4^\alpha \sum_{k+l+m=0} |k|^{2\alpha} e^{\sigma|k|} |\hat{u}(k)| \cdot e^{\sigma|l|} |l| \cdot |\hat{v}(l)| |\hat{w}(m)| + 8\pi^3 4^\alpha \sum_{k+l+m=0} e^{\sigma|k|} |\hat{u}(k)| \cdot |l|^{2\alpha+1} e^{\sigma|l|} \cdot |\hat{v}(l)| |\hat{w}(m)|. \]

Below, \( e^{\sigma A^{1/2}} u_* = e^{\sigma(-\Delta)^{1/2}} u_* \). Rewrite
\[ I \leq 8\pi^3 4^\alpha \left| \int_\Omega ((-\Delta)^\alpha e^{\sigma A^{1/2}} u_*) \cdot ((-\Delta)^{1/2} e^{\sigma A^{1/2}} v_*) \cdot w_* \, dx \right| + 8\pi^3 4^\alpha \left| \int_\Omega (e^{\sigma A^{1/2}} u_*) \cdot ((-\Delta)^{\alpha+1/2} e^{\sigma A^{1/2}} v_*) \cdot w_* \, dx \right| \overset{\text{def}}{=} 8\pi^3 4^\alpha (l_1 + l_2). \]
We recall the Sobolev and Agmon inequalities:

\[\| u_* \|_{L^6(\Omega)} \leq c_1 |(-\Delta)^{1/2} u_*|, \]
\[\| u_* \|_{L^\infty(\Omega)} \leq c_2 |(-\Delta)^{1/2} u_*|^{1/2} |(-\Delta) u_*|^{1/2}.\]

- For \( I_1 \):
  \[
  I_1 \leq \| (-\Delta)^\alpha e^{\sigma A^{1/2}} u_* \|_{L^3(\Omega)} \| (-\Delta)^{1/2} e^{\sigma A^{1/2}} v_* \|_{L^6(\Omega)} |w_*| \\
  \leq C \| (-\Delta)^\alpha e^{\sigma A^{1/2}} u_* \|_{L^6(\Omega)} \| (-\Delta)^{1/2} e^{\sigma A^{1/2}} v_* \|_{L^6(\Omega)} |w_*| \\
  \leq C |(-\Delta)^{\alpha+1/2} e^{\sigma A^{1/2}} u_* |(-\Delta)e^{\sigma A^{1/2}} v_* |w_*| \\
  \leq C |A^{\alpha+1/2} e^{\sigma A^{1/2}} u||Ae^{\sigma A^{1/2}} v||w|.\]

- For \( I_2 \):
  \[
  I_2 \leq \| e^{\sigma A^{1/2}} u_* \|_{L^\infty(\Omega)} |(-\Delta)^{\alpha+1/2} e^{\sigma A^{1/2}} v_* |w_*| \\
  \leq C |(-\Delta)^{1/2} e^{\sigma A^{1/2}} u_*|^{1/2} |(-\Delta)e^{\sigma A^{1/2}} u_* |^{1/2} \cdot |(-\Delta)^{\alpha+1/2} e^{\sigma A^{1/2}} v_* |w_*| \\
  \leq C |A^{1/2} e^{\sigma A^{1/2}} u|^{1/2} |Ae^{\sigma A^{1/2}} u|^{1/2} |A^{\alpha+1/2} e^{\sigma A^{1/2}} v||w|.\]
For $\alpha \geq 1/2$, we have $\alpha + 1/2 \geq 1 > 1/2$. It follows that

$$|\langle A^\alpha e^{\sigma A^{1/2}} B(u, v), w \rangle| \leq C |A^{\alpha+1/2} e^{\sigma A^{1/2}} u| |A^{\alpha+1/2} e^{\sigma A^{1/2}} v||w|.$$ 

Hence,

$$|A^\alpha e^{\sigma A^{1/2}} B(u, v)| \leq C |A^{\alpha+1/2} e^{\sigma A^{1/2}} u| |A^{\alpha+1/2} e^{\sigma A^{1/2}} v|.$$
Proposition

Let $\delta \in (0, 1)$, $\lambda \in (1 - \delta, 1]$ and $\sigma \geq 0$, $\alpha \geq 1/2$. There are $C_0$, $C_1 > 0$ such that if

$$|A^\alpha u^0| \leq C_0, \quad |f(t)|_{\alpha-1/2,\sigma} \leq C_1 e^{-\lambda t}, \quad \forall t \geq 0,$$

then there exists a unique solution $u \in C([0, \infty), \mathcal{D}(A^\alpha))$ that satisfies and

$$|u(t)|_{\alpha,\sigma} \leq \sqrt{2}C_0 e^{-(1-\delta)t}, \quad \forall t \geq t_*,$$

where $t_* = 6\sigma/\delta$. Moreover, one has for all $t \geq t_*$ that

$$\int_t^{t+1} |u(\tau)|_{\alpha+1/2,\sigma}^2 d\tau \leq \frac{2C_0^2}{1-\delta} e^{-2(1-\delta)t}.$$
Proof: Estimates of the Gevrey norms.

Let $\varphi(t)$ be a function in $C^\infty(\mathbb{R})$ such that

$$
\varphi((\infty, 0]) = \{0\}, \quad \varphi([0, t_*]) = [0, \sigma], \quad \varphi([t_*, \infty)) = \{\sigma\},
$$

and

$$
0 < \varphi'(t) < 2\sigma/t_* = \delta/3 \quad \text{for all } t \in (0, t_*).
$$

From equation, we have

$$
\frac{d}{dt}(A^\alpha e^{\varphi(t)A^{1/2}} u(t)) = A^\alpha e^{\varphi(t)A^{1/2}} u' + \varphi'(t)A^{1/2}A^\alpha e^{\varphi(t)A^{1/2}} u
$$

$$
= A^\alpha e^{\varphi(t)A^{1/2}} (-Au - B(u, u) + f) + \varphi'(t)A^{1/2}A^\alpha e^{\varphi(t)A^{1/2}} u.
$$
Taking inner product of the equation with $A^\alpha e^{\varphi(t)}A^{1/2} u(t)$ gives

$$\frac{1}{2} \frac{d}{dt} |u|_{\alpha,\varphi(t)}^2 + |A^{1/2} u|_{\alpha,\varphi(t)}^2 = \varphi'(t) \langle A^{2\alpha+1/2} e^{2\varphi(t)}A^{1/2} u, u \rangle$$

$$- \langle A^\alpha e^{\varphi(t)}A^{1/2} B(u, u), A^\alpha e^{\varphi(t)}A^{1/2} u \rangle + \langle A^{\alpha-1/2} e^{\varphi(t)}A^{1/2} f, A^{\alpha+1/2} e^{\varphi(t)}A^{1/2} u \rangle.$$

Then

$$\frac{1}{2} \frac{d}{dt} |u|_{\alpha,\varphi(t)}^2 + |A^{1/2} u|_{\alpha,\varphi(t)}^2$$

$$\leq \varphi'(t) |u|_{\alpha+1/2,\varphi(t)}^2 + K^\alpha |A^{1/2} u|_{\alpha,\varphi(t)}^2 |u|_{\alpha,\varphi(t)} + |f(t)|_{\alpha-1/2,\varphi(t)} |u|_{\alpha+1/2,\varphi(t)}$$

$$\leq \frac{\delta}{3} |u|_{\alpha+1/2,\varphi(t)}^2 + K^\alpha |A^{1/2} u|_{\alpha,\varphi(t)}^2 |u|_{\alpha,\varphi(t)} + \frac{3}{4\delta} |f(t)|_{\alpha-1/2,\varphi(t)}^2 + \frac{\delta}{3} |u|_{\alpha+1/2}^2.$$

This implies

$$\frac{1}{2} \frac{d}{dt} |u|_{\alpha,\varphi(t)}^2 + \left(1 - \frac{2\delta}{3} - K^\alpha |u|_{\alpha,\varphi(t)}\right) |A^{1/2} u|_{\alpha,\varphi(t)}^2 \leq \frac{3}{4\delta} |f(t)|_{\alpha-1/2,\varphi(t)}^2.$$
Let \( T \in (0, \infty) \). Note that \(|u(0)|_{\alpha,\varphi(0)} = |A^\alpha u^0| < 2C_0\). Assume that
\[ |u(t)|_{\alpha,\varphi(t)} \leq 2C_0, \quad \forall t \in [0, T). \]

Then for \( t \in (0, T) \), we have
\[ \frac{d}{dt} |u|_{\alpha,\varphi(t)}^2 + 2(1 - \delta)|A^{1/2}u|_{\alpha,\varphi(t)}^2 \leq \frac{3}{2\delta} |f(t)|_{\alpha-1/2,\sigma}^2 \leq \frac{3C_1^2}{2\delta} e^{-2\lambda t}. \]

Applying Gronwall’s inequality yields for all \( t \in (0, T) \) that
\[ |u(t)|_{\alpha,\varphi(t)}^2 \leq e^{-2(1-\delta)t} |u^0|_{\alpha,0}^2 + \frac{3C_1^2}{2\delta} e^{-2(1-\delta)t} \int_0^t e^{2(1-\delta)\tau} \cdot e^{-2\lambda \tau} d\tau \]
\[ \leq e^{-2(1-\delta)t} |u^0|_{\alpha,0}^2 + \frac{3C_1^2}{4\delta(\lambda - 1 + \delta)} e^{-2(1-\delta)t} \]
\[ = \left( |u^0|_{\alpha,0}^2 + C_0^2 \right) e^{-2(1-\delta)t}. \]

We obtain
\[ |u(t)|_{\alpha,\varphi(t)}^2 \leq 2C_0^2 e^{-2(1-\delta)t}, \]
which gives
\[ |u(t)|_{\alpha,\varphi(t)} \leq \sqrt{2C_0} e^{-(1-\delta)t}, \quad \forall t \in (0, T). \]
• In particular, letting \( t \to T^- \) yields

\[
\lim_{t \to T^-} |u(t)|_{\alpha, \varphi(t)} \leq \sqrt{2}C_0 < 2C_0.
\]

• By the standard contradiction argument, we have that the inequality holds for all \( t > 0 \)

• Since \( \varphi(t) = \sigma \) for \( t \geq t^* \),

\[
|u(t)|_{\alpha, \sigma} \leq \sqrt{2}C_0 e^{-(1-\delta)t}, \quad \forall t \geq 0.
\]
For any $t \geq 0$,

$$|u(t)|^2 \leq Ce^{-t},$$

Also,

$$|u(t)|^2 + \int_{t_0}^{t} \|u(\tau)\|^2 d\tau \leq |u(t_0)|^2 + \int_{t_0}^{t} |f(\tau)|^2 d\tau,$$

for $t_0 = 0$ and almost all $t_0 \in (0, \infty)$, and all $t \geq t_0$. Consequently,

$$\int_{t}^{t+1} \|u(\tau)\|^2 d\tau \leq Ce^{-t}.$$
For $\alpha \in [0, \infty)$ and $\delta \in (0, 1)$, there exists a positive number $T_0$ such that

$$|u(T_0 + t)|_{\alpha, \sigma_0} \leq e^{-(1-\delta)t} \quad \forall t \geq 0,$$

and

$$|B(u(T_0 + t), u(T_0 + t))|_{\alpha, \sigma_0} \leq e^{-2(1-\delta)t} \quad \forall t \geq 0.$$  

Note: Can use different bootstrapping procedures for $\sigma_0 > 0$ (faster) and $\sigma_0 = 0$ (gradually).
Proof of Asymptotic Expansion. First step $N = 1$

Recall

\[ f(t) \sim \sum_{n=1}^{\infty} f_n(t)e^{-nt} = \sum_{n=1}^{\infty} F_n(t). \]

Need to prove

\[ u(t) \sim \sum_{n=1}^{\infty} q_n(t)e^{-nt}. \]

Let $w_0(t) = e^t u(t)$ and $w_{0,k}(t) = R_k w_0(t)$. We have

\[ \frac{d}{dt} w_0 + (A - 1) w_0 = f_1 + H_1(t), \]

where

\[ H_1(t) = e^t(f - F_1 - B(u, u)). \]

Taking the projection $R_k$ gives

\[ \frac{d}{dt} w_{0,k} + (k - 1) w_{0,k} = R_k f_1 + R_k H_1(t). \]

Note that $R_k f_1(t)$ is a polynomial in $R_k H$. 

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Lemma

Let \((X, \| \cdot \|)\) be a Banach space. Suppose \(y(t)\) is in \(C([0, \infty), X)\) and \(C^1((0, \infty), X)\) that solves the following ODE

\[
\frac{dy}{dt} + \alpha y = p(t) + g(t) \quad \text{for } t > 0,
\]

where constant \(\alpha \in \mathbb{R}\), \(p(t)\) is a \(X\)-valued polynomial in \(t\), and \(g(t) \in C([0, \infty), X)\) satisfies

\[
\|g(t)\| \leq Me^{-\delta t} \quad \forall t \geq 0, \quad \text{for some } M, \delta > 0.
\]

Define \(q(t)\) for \(t \in \mathbb{R}\) by

\[
q(t) = \begin{cases} 
  e^{-\alpha t} \int_{-\infty}^{t} e^{\alpha \tau} p(\tau) d\tau & \text{if } \alpha > 0, \\
  y(0) + \int_0^{\infty} g(\tau) d\tau + \int_0^{t} p(\tau) d\tau & \text{if } \alpha = 0, \\
  -e^{-\alpha t} \int_{t}^{\infty} e^{\alpha \tau} p(\tau) d\tau & \text{if } \alpha < 0.
\end{cases}
\]
Then \( q(t) \) is an \( X \)-valued polynomial that satisfies

\[
\frac{dq(t)}{dt} + \alpha q(t) = p(t) \quad \forall t \in \mathbb{R},
\]

and the following estimates hold.

(i) If \( \alpha > 0 \) then

\[
\|y(t) - q(t)\| \leq \left( \|y(0) - q(0)\| + \frac{M}{|\alpha - \delta|} \right) e^{-\min\{\delta, \alpha\}t}, \quad t \geq 0, \text{ for } \alpha \neq \delta,
\]

and

\[
\|y(t) - q(t)\| \leq (\|y(0) - q(0)\| + Mt) e^{-\delta t}, \quad t \geq 0, \text{ for } \alpha = \delta.
\]

(ii) If \( (\alpha = 0) \) or \( (\alpha < 0 \text{ and } \lim_{t \to \infty} e^{\alpha t} y(t) = 0) \) then

\[
\|y(t) - q(t)\| \leq \frac{Me^{-\delta t}}{|\alpha - \delta|} \quad \forall t \geq 0.
\]
For the Lemma, we just use the following elementary identities: for $\beta > 0$, integer $d \geq 0$, and any $t \in \mathbb{R}$,

\[
\int_{-\infty}^{t} \tau^d e^{\beta \tau} \, d\tau = e^{\beta t} \sum_{n=0}^{d} \frac{(-1)^{d-n} d!}{n! \beta^{d+1-n}} t^n,
\]

\[
\int_{t}^{\infty} \tau^d e^{-\beta \tau} \, d\tau = e^{-\beta t} \sum_{n=0}^{d} \frac{d!}{n! \beta^{d+1-n}} t^n.
\]

N=1 (continued). Then there exists a polynomial $q_1(t)$ such that

\[
|w_0(t) - q_1(t)|_{\alpha,\sigma_0} = O(e^{-\delta t}).
\]

Hence

\[
|u(t) - q_1(t)e^{-t}|_{\alpha,\sigma_0} = O(e^{-(1+\delta)t}).
\]
Induction step

Denote $\varepsilon_* \in (0, \delta^*_{N+1})$ and $\bar{u}_N(t) = \sum_{n=1}^{N} u_n(t)$. 

Remainder $v_N(t) = u(t) - \bar{u}_N(t)$ satisfies for any $\beta > 0$ that

$$|v_N(t)|_{\beta, \sigma_0} = O(e^{-(N+\varepsilon_*)t}) \text{ as } t \to \infty.$$ 

**Evolution of $v_N$:**

$$\frac{d}{dt} v_N + Av_N + \sum_{m+j=N+1} B(u_m, u_j) = F_{N+1}(t) + h_N(t),$$

where

$$h_N(t) = -B(v_N, u) - B(\bar{u}_N, v_N) - \sum_{1 \leq m, j \leq N \atop m+j \geq N+2} B(u_m, u_j) + \tilde{F}_{N+1}(t),$$

$$\tilde{F}_{N+1}(t) = f(t) - \sum_{n=1}^{N+1} F_n(t).$$

Fact:

$$h_N(t) = O_{\alpha, \sigma_0}(e^{-(N+1+\varepsilon_*)t}).$$
Let $w_N(t) = e^{(N+1)t}v_N(t)$ and $w_{N,k} = R_k w_N(t)$. The ODE for $w_{N,k}$:

$$\frac{d}{dt}w_{N,k} + (k - (N + 1))w_{N,k} + \sum_{m+j=N+1} R_k B(q_m, q_j) = R_k f_{N+1} + H_{N,k},$$

with $H_{N,k} = e^{(N+1)t} R_k h_N(t)$.

Fact:

$$|H_{N,k}|_{\alpha,\sigma_0} = O(e^{-\varepsilon_* t}).$$

Then there are $T > T_0$ and $M > 0$ such that for $t \geq 0$

$$|H_{N,k}(T + t)|_{\alpha,\sigma_0} \leq M e^{-\varepsilon_* t}.$$
Case $k = N + 1$

By Lemma(ii), there is a polynomial $q_{N+1,N+1}(t)$ valued in $R_{N+1}H$ such that

$$|w_{N,N+1}(T + t) - q_{N+1,N+1}(t)|_{\alpha,\sigma_0} = \mathcal{O}(e^{-\varepsilon^* t}),$$

thus,

$$|R_{N+1}w_N(t) - q_{N+1,N+1}(t - T)|_{\alpha,\sigma_0} = \mathcal{O}(e^{-\varepsilon^* t}).$$
Case $k \leq N$

Note

$$\lim_{t \to \infty} e^{(k-(N+1))t} w_{N,k}(t) = \lim_{t \to \infty} e^{kt} R_k v_N(t) = 0.$$ 

Applying Lemma(ii) with $\alpha = k - N - 1 < 0$, there is a polynomial $q_{N+1,k}(t)$ valued in $R_k H$ such that

$$|w_{N,k}(T + t) - q_{N+1,k}(t)|_{\alpha,\sigma_0} = O(e^{-\epsilon_* t}),$$

$$|R_k w_N(t) - q_{N+1,k}(t - T)|_{\alpha,\sigma_0} = O(e^{-\epsilon_* t}).$$
Case $k \geq N + 2$

Similarly, applying Lemma(i), there is a polynomial $q_{N+1,k}(t)$ valued in $R_k H$ such that

$$
|w_{N,k}(T + t) - q_{N+1,k}(t)|_{\alpha,\sigma_0} \leq \left( |R_k v_N(T)|_{\alpha,\sigma_0} + |q_{N+1,k}(0)|_{\alpha,\sigma_0} + \frac{M}{k - (N + 1)} \right) e^{-\epsilon_* t}.
$$

Thus

$$
|R_k w_N(t) - q_{N+1,k}(t - T)|_{\alpha,\sigma_0} \leq e^{\epsilon_* T} \left( |R_k v_N(T)|_{\alpha,\sigma_0} + |q_{N+1,k}(0)|_{\alpha,\sigma_0} + \frac{M}{k - (N + 1)} \right) e^{-\epsilon_* t}.
$$
Define $q_{N+1}(t) = \sum_{k=1}^{\infty} q_{N+1,k}(t - T)$. Then squaring and summing in $k$, we obtain

$$\sum_{k=N+2}^{\infty} |R_k w_N(t) - q_{N+1,k}(t - T)|^2_{\alpha,\sigma_0} \leq 3e^{2\varepsilon^* T} \left( \sum_{k=N+2}^{\infty} |R_k v_N(T)|^2_{\alpha,\sigma_0} + \sum_{k=N+2}^{\infty} |R_k q_{N+1}(T)|^2_{\alpha,\sigma_0} + \sum_{k=N+2}^{\infty} \frac{M^2}{(k - (N + 1))^2} e^{-2\varepsilon^* t} \right)$$

$$= \mathcal{O}(e^{-2\varepsilon^* t}).$$

Thus,

$$|w_N(t) - q_{N+1}(t)|_{\alpha,\sigma_0} = \mathcal{O}(e^{-\varepsilon^* t}),$$

therefore,

$$|v_N(t) - e^{-(N+1)t} q_{N+1}(t)|_{\alpha,\sigma_0} = \mathcal{O}(e^{-(N+1+\varepsilon^*) t}).$$
Check ODE for $u_{N+1}(t)$

The polynomial $q_{N+1}(t)$ satisfies

$$\frac{d}{dt} R_k q_{N+1}(t) + (k-(N+1)) R_k q_{N+1}(t) + \sum_{m+j=N+1} R_k B(q_m, q_j) = R_k f_{N+1}(t),$$

$$\frac{d}{dt} R_k u_{N+1}(t) + kR_k u_{N+1}(t) + \sum_{m+j=N+1} R_k B(u_m, u_j) = R_k F_{N+1}(t) \quad \forall k \geq 1,$$

which we rewrite as

$$\frac{d}{dt} u_{N+1}(t) + A u_{N+1}(t) + \sum_{m+j=N+1} B(u_m, u_j) = F_{N+1}(t).$$
THANK YOU FOR YOUR ATTENTION.