Asymptotic expansions in large time for solutions of non-autonomous differential equations

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Outline

1. Introduction

2. Main results
   - I. Exponentially decaying forces
   - II. Power-decaying forces

3. Sketch of proofs
   - I. Case of exponential decay
   - II. Case of power decay

4. Application to solutions near special periodic orbits
1. Introduction
Functional form of NSE

\[ \frac{du}{dt} + Au + B(u, u) = f(t), \]

where \( A \) is the (unbounded) Stokes operator with, after scaling, \( \sigma(A) \subset \mathbb{N} \). Note: quadratic nonlinearity.

When \( f = 0 \), solution \( u(t) \) admits an expansion

\[ u(t) \sim \sum_{n=1}^{\infty} q_n(t)e^{-nt}, \quad \text{with polynomials } q_n(t), \]

meaning

\[ \| u(t) - \sum_{n=1}^{N} q_n(t)e^{-nt} \| = \mathcal{O}(e^{-(N+\varepsilon)t}) \quad ast \rightarrow \infty. \]
Extension to time-dependent forces

NSE with periodic boundary conditions.

  \[ f(t) \sim \sum_{n=1}^{\infty} f_n(t)e^{-nt}. \]

Same expansion for \( u(t) \):

\[ u(t) \sim \sum_{n=1}^{\infty} q_n(t)e^{-nt}. \]

Note: exponential rates are in the additive semigroup generated by \( \sigma(A) \).

  \[ f(t) \sim \sum_{n=1}^{\infty} \phi_n t^{-n}. \]

Then

\[ u(t) \sim \sum_{n=1}^{\infty} \xi_n t^{-n}. \]
Our problems

Focus on ordinary differential equations (ODE) in $\mathbb{R}^n$:

$$\frac{dy}{dt} = -Ay + G(y) + f(t), \quad y(0) = y_0,$$

where

- unknown $y(t) \in \mathbb{R}^n$, given initial condition $y_0 \in \mathbb{R}^n$,
- $A$ is an $n \times n$ matrix,
- $G(y)$ locally is Lipschitz, and has expansion
  $$G(y) \sim \sum_{m=2}^{\infty} \mathcal{L}_m(y) \text{ as } y \to 0,$$
  where each $\mathcal{L}_m : \mathbb{R}^n \to \mathbb{R}^n$ is a homogeneous polynomial of degree $m$,
- $f(t)$ decays exponentially or algebraically at any rates.

**Goal:** Obtain asymptotic expansions for solutions $y(t)$ as $t \to \infty$. 
Assumption 1

• Matrix $A$ has positive eigenvalues

$$\Lambda_1 \leq \Lambda_2 \leq \ldots \leq \Lambda_n,$$

and the corresponding eigenvectors form a basis of $\mathbb{R}^n$.

• Rewrite the spectrum

$$\sigma(A) = \{\Lambda_k : k = 1, 2, \ldots, n\} = \{\lambda_1 < \lambda_2 < \ldots\}.$$
Assumption 2

- Rewrite the homogeneous polynomials as

\[ \mathcal{L}_m(y) = L_m(y, y, \ldots, y) \quad (m \text{ times}), \]

where \( L_m : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}^n \) is an \( m \)-linear mapping.

- For each \( N \geq 2 \),

\[ |G(y) - \sum_{m=2}^{N} \mathcal{L}_m(y)| = O(|y|^{N+\varepsilon}) \text{ as } y \to 0, \]

for some \( \varepsilon > 0 \).
Global existence

**Theorem**

There exists $\varepsilon_0 > 0$ such that if

$$|y_0| < \varepsilon_0, \quad \|f\|_\infty = \sup_{t \geq 0} |f(t)| < \varepsilon_0,$$

then there exists a solution $y(t)$ on $[0, \infty)$.

In addition, if

$$\lim_{t \to \infty} f(t) = 0,$$

then

$$\lim_{t \to \infty} y(t) = 0.$$

Note: for small $y$: $|G(y)| \leq C|y|^2$.

Throughout, we consider global solution $y(t)$ on $[0, \infty)$ that converges to zero as $t \to \infty$. 
2. Main results

- I. Exponentially decaying forces
- II. Power-decaying forces
I. Exponentially decaying forces

Notation. Exponential expansion (in time):

\[ y(t) \sim \sum_{k=1}^{\infty} p_k(t)e^{-\alpha_k t}, \]

where \( \alpha_k > 0 \) are strictly increasing constants, and \( p_k \) are polynomials, if for any \( N \geq 1 \), there exists \( \varepsilon > 0 \), such that

\[ |y(t) - \sum_{k=1}^{N} p_k(t)e^{-\alpha_k t}| = \mathcal{O}(e^{-(\alpha N + \varepsilon)t}) \quad \text{as } t \to \infty. \]
Assumption

Force

\[ f(t) \exp \sim \sum_{k=1}^{\infty} \tilde{p}_k(t)e^{-\alpha_k t}. \]

Let \( S \) be the additive semigroup generated by \( \lambda_k \) and \( \alpha_k \).

Re-order:

\[ S = \{ \mu_1 < \mu_2 < \mu_3 < \ldots \}. \]

Re-write

\[ f(t) \exp \sim \sum_{k=1}^{\infty} p_k(t)e^{-\mu_k t} = \sum_{k=1}^{\infty} f_k(t). \]

For \( \mu \in S \), denote \( R_\mu = R_\lambda \) the projection if \( \lambda \in \sigma(A) \), otherwise, \( R_\mu = 0 \).

Still have

\[ AR_\mu y = \mu R_\mu y, \quad \mathbb{R}^n = \bigoplus_{k=1}^{\infty} R_{\mu_k}. \]
Theorem (Cao-H.)

For solution \( y(t) \), there exist vector-valued polynomials \( q_n(t) \) such that

\[
y(t) \sim \sum_{k=1}^{\infty} q_k(t)e^{-\mu_k t} \quad \text{as } t \to \infty.
\]

In fact, the polynomials \( q_k(t) \)'s solve the linear systems

\[
q'_k = -(A-\mu_k)q_k + \sum_{m=2}^{N_k} \sum_{\mu_j, m_1 + \mu_j, m_2 + \cdots + \mu_j, m = \mu_k} L_m(q_{j, m_1}, q_{j, m_2}, \ldots, q_{j, m}) + p_k(t).
\]

Equivalently, \( y_k(t) = q_k(t)e^{-\mu_k t} \) solve

\[
y'_k = -Ay_k + \sum_{m=2}^{N_k} \sum_{\mu_j, m_1 + \mu_j, m_2 + \cdots + \mu_j, m = \mu_k} L_m(y_{j, m_1}, y_{j, m_2}, \ldots, y_{j, m}) + f_k(t).
\]
Remarks.

- In the sums above, $1 \leq j_m, \ell \leq k - 1$, and $N_k$ is finite depending on $k$, and sufficiently large, for e.g., $N_k \mu_1 \geq \mu_k$.
- Each ODE is a linear system, with the forcing term defined by previous steps.
- The $q_k$’s are unique polynomial solutions provided $R_{\mu_k}q_k(0)$ is given.
- In autonomous case ($f = 0$),

$$q'_k = -(A - \mu_k)q_k + \sum_{m=2}^{N_k} \sum_{\mu_{jm,1} + \mu_{jm,2} + \ldots + \mu_{jm,m} = \mu_k} L_m(q_{jm,1}, q_{jm,2}, \ldots, q_{jm,m}).$$

Compare this with non-autonomous case.
- The $q_k$’s depend on the initial data $y_0$. 
Notation. Power expansion (in time):

\[ y(t)^{\text{pow.}} \sim \sum_{j=1}^{\infty} \xi_j t^{-\alpha_j}, \]

where \( \alpha_j > 0 \) are strictly increasing, and \( \xi_j \in \mathbb{R}^n \) are constant vectors, if for any \( N \geq 1 \), there exists \( \varepsilon > 0 \), such that

\[ |y(t) - \sum_{j=1}^{N} \xi_j t^{-\alpha_j}| = O(t^{-(\alpha_N + \varepsilon)}) \quad \text{as} \ t \to \infty. \]
Assumption

The force has the expansion

$$f(t) \overset{\text{pow.}}{\sim} \sum_{k=1}^{\infty} \tilde{\eta}_k t^{-\alpha_k},$$

where $\tilde{\eta}_k \in \mathbb{R}^n$, and

$$0 < \alpha_1 < \alpha_2 < \ldots$$

Let $S = (\text{additive semigroup generated } \alpha_k \text{'s}) + (\mathbb{N} \cup \{0\})$.

Denote

$$S = \{0 < \mu_1 = \alpha_1 < \mu_2 < \mu_3 < \ldots\}.$$ 

Rewrite

$$f(t) \overset{\text{pow.}}{\sim} \sum_{k=1}^{\infty} \eta_k t^{-\mu_k} = \sum_{k=1}^{\infty} f_k(t).$$
Theorem (Cao-H.)

For any solution $y(t)$, ones have

$$y(t) \sim \sum_{k=1}^{\infty} \xi_k t^{-\mu_k} \quad \text{as } t \to \infty,$$

where constant vectors $\xi_k \in \mathbb{R}^n$ satisfy

$$A \xi_k = \sum_{m=2} \sum_{\mu_{j,1} + \mu_{j,2} + \cdots + \mu_{j,m} = \mu_k} L_m(\xi_{j,1}, \xi_{j,2}, \ldots, \xi_{j,m}) + \eta_k + \xi_p \mu_p$$

in case there exists $1 \leq p \leq k - 1$ such that $\mu_p + 1 = \mu_k$; or

$$A \xi_k = \sum_{m=2} \sum_{\mu_{j,1} + \mu_{j,2} + \cdots + \mu_{j,m} = \mu_k} L_m(\xi_{j,1}, \xi_{j,2}, \ldots, \xi_{j,m}) + \eta_k,$$

in case $\mu_p + 1 \neq \mu_k$ for all $1 \leq p \leq k - 1$. 
The $\xi_k$’s and hence the expansion are independent on initial data $y_0$, contrasting with the exponential case.

It means that all (decaying) solutions have the same power expansion.
Example

Assume:

- $\alpha_k = k$ for all $k \in \mathbb{N}$
- $G(y) = B(y, y)$.

Then $\mu_k = k$, and $S = \mathbb{N}$. Expansion

$$y(t) \sim \sum_{k=1}^{\infty} \xi_k t^{-k},$$

where

$$\xi_1 = A^{-1} \eta_1,$$

and for $k \geq 2$,

$$\xi_k = A^{-1} \left\{ (k - 1)\xi_{k-1} + \sum_{j=1}^{k-1} B(\xi_j, \xi_{k-j}) + \eta_k \right\}.$$
3. Sketch of proofs

I. Case of exponential decay
II. Case of power decay
I. Case of exponential decay

Recall

\[ f(t) \exp. \sum_{k=1}^{\infty} p_k(t)e^{-\mu_k t} = \sum_{k=1}^{\infty} f_k(t). \]

Need to prove

\[ y(t) \exp. \sum_{k=1}^{\infty} q_k(t)e^{-\mu_k t}. \]
Induction step.

Let $y_k(t) = q_k(t)e^{-\mu_k t}$, for $1 \leq k \leq N$, $\bar{y}_N = \sum_{k=1}^{N} y_k$ and $v_N = y - \bar{y}_N$. Induction hypotheses: for $k = 1, 2, \ldots, N$

$$v_k = O(e^{-\mu_k - \delta_k} t),$$

and equations for $y_k$’s hold true for $k = 1, 2, \ldots, N$. We will construct the polynomial $q_{N+1}(t)$ such that

$$|w_N(t) - q_{N+1}(t)| = O(e^{-\delta_{N+1} t}),$$

where

$$w_N(t) = e^{\mu_{N+1} t} v_N(t).$$
Equation for $w_N(t)$:

\[
\begin{align*}
 w_N' &= -(A - \mu_{N+1})w_N \\
 &\quad + \sum_{m \geq 2} \sum_{\mu_{j_1} + \mu_{j_2} + \cdots + \mu_{j_m} = \mu_{N+1}} L_m(q_{j_1}, q_{j_2}, \cdots, q_{j_m}) \\
 &\quad + \mathcal{O}(e^{-\delta t}).
\end{align*}
\]

For $\mu \in S$, taking $R_\mu$ of the equation gives

\[
\begin{align*}
 (R_\mu w_N)' &= -(\mu - \mu_{N+1})R_\mu w_N \\
 &\quad + \sum_{m \geq 2} \sum_{\mu_{j_1} + \mu_{j_2} + \cdots + \mu_{j_m} = \mu_{N+1}} R_\mu L_m(q_{j_1}, q_{j_2}, \cdots, q_{j_m}) \\
 &\quad + \mathcal{O}(e^{-\delta t}).
\end{align*}
\]
Lemma

Let \((X, \| \cdot \|)\) be a Banach space. Suppose \(y(t)\) is in \(C([0, \infty), X)\) and \(C^1((0, \infty), X)\) that solves the following ODE

\[
\frac{dy}{dt} + \alpha y = p(t) + g(t) \quad \text{for } t > 0,
\]

where constant \(\alpha \in \mathbb{R}\), \(p(t)\) is a \(X\)-valued polynomial in \(t\), and \(g(t) \in C([0, \infty), X)\) satisfies

\[
\|g(t)\| \leq Me^{-\delta t} \quad \forall t \geq 0, \quad \text{for some } M, \delta > 0.
\]

Define \(q(t)\) for \(t \in \mathbb{R}\) by

\[
q(t) = \begin{cases} 
  e^{-\alpha t} \int_{-\infty}^{t} e^{\alpha \tau} p(\tau) d\tau & \text{if } \alpha > 0, \\
  y(0) + \int_{0}^{\infty} g(\tau) d\tau + \int_{0}^{t} p(\tau) d\tau & \text{if } \alpha = 0, \\
  -e^{-\alpha t} \int_{t}^{\infty} e^{\alpha \tau} p(\tau) d\tau & \text{if } \alpha < 0.
\end{cases}
\]
Then $q(t)$ is an $\mathcal{X}$-valued polynomial that satisfies

$$\frac{dq(t)}{dt} + \alpha q(t) = p(t) \quad \forall t \in \mathbb{R},$$

and the following estimates hold.

(i) If $\alpha > 0$ then

$$\|y(t) - q(t)\| \leq \left(\|y(0) - q(0)\| + \frac{M}{|\alpha - \delta|}\right) e^{-\min\{\delta, \alpha\} t}, \quad t \geq 0, \text{ for } \alpha \neq \delta,$$

and

$$\|y(t) - q(t)\| \leq (\|y(0) - q(0)\| + Mt)e^{-\delta t}, \quad t \geq 0, \text{ for } \alpha = \delta.$$

(ii) If $(\alpha = 0)$ or $(\alpha < 0$ and $\lim_{t \to \infty} e^{\alpha t}y(t) = 0)$ then

$$\|y(t) - q(t)\| \leq \frac{Me^{-\delta t}}{|\alpha - \delta|} \quad \forall t \geq 0.$$
Applying the above ODE lemma, then there exists polynomial $q_{N+1,j} \in R_{\mu_j}(\mathbb{R}^n)$ such that

$$|R_{\mu_j}w_N(t) - q_{N+1,j}(t)| = O(e^{-\delta_{N+1}t}),$$

Define $q_{N+1} = \sum_j q_{N+1,j}$ (finite sum). Then

$$|w_N(t) - q_{N+1}(t)| = O(e^{-\delta_{N+1}t}),$$

which yields

$$|y(t) - \sum_{k=1}^{N} q_k(t)e^{-\mu_k t} - q_{N+1}(t)e^{-\mu_{N+1} t}| = O(e^{-(\mu_{N+1}+\delta_{N+1})t}).$$
Recall

\[ f(t) \overset{\text{pow.}}{\sim} \sum_{k=1}^{\infty} \eta_k t^{-\mu_k}. \]

Need to prove

\[ y(t) \overset{\text{pow.}}{\sim} \sum_{k=1}^{\infty} \xi_k t^{-\mu_k}. \]

**Induction step.** Let \( y_k = \xi_k t^{-\mu_k}, \bar{y}_N = \sum_{k=1}^{N} y_k \) and \( v_N = y - \bar{y}_N \).

Suppose

\[ |v_N| = O(t^{-(\mu_N + \delta_N)}). \]

Let

\[ w_N = t^{\mu_N+1} v_N. \]
Induction step.

Equation for $w_N(t)$:

$$w_N' = -Aw_N$$

$$+ t^{\mu_{N+1}} \left\{ t^{-\mu_{N+1}} \sum_{m \geq 2} \sum_{\mu_{jm,1} + \mu_{jm,2} + \cdots + \mu_{jm,m} = \mu_{N+1}} L_m(\xi_{jm,1}, \xi_{jm,2}, \ldots, \xi_{jm,m}) \right\}$$

$$+ \eta_{N+1} t^{-\mu_{N+1}}$$

$$+ \sum_{k=1}^{N} \left( t^{-\mu_k} \sum_{m \geq 2} \sum_{\mu_{jm,1} + \mu_{jm,2} + \cdots + \mu_{jm,m} = \mu_k} L_m(\xi_{jm,1}, \xi_{jm,2}, \ldots, \xi_{jm,m}) \right)$$

$$- A\xi_k t^{-\mu_k} + \eta_k t^{-\mu_k} \right) + \sum_{p=1}^{N} \mu_p \xi_p t^{-\left(\mu_p + 1\right)} \right\} + O(t^{-\delta}).$$
Note $\mu_N + 1 \geq \mu_{N+1}$. Moreover

$$\{\mu_p + 1 : 1 \leq p \leq N - 1\} \cap [\mu_1, \mu_{N+1}) \subset \{\mu_k : 1 \leq k \leq N\}.$$ 

Then distribute the red sum into the others including possible $O(t^{-\delta})$ gives

$$w_N' = -Aw_N$$

$$+ t^{\mu_{N+1}} \left\{ t^{-\mu_{N+1}} \sum_{m \geq 2} \sum_{\mu_{j_{m,1}} + \mu_{j_{m,2}} + \cdots \mu_{j_{m,m}} = \mu_{N+1}} L_m(\xi_{j_{m,1}}, \xi_{j_{m,2}}, \ldots, \xi_{j_{m,m}}) \right\}$$

$$+ \eta_{N+1} t^{-\mu_{N+1}} + \mu_p \xi_p t^{-(\mu_p + 1)} \right|_{\mu_p + 1 = \mu_{N+1}}$$

$$+ \sum_{k=1}^{N} \left\{ -A\xi_k t^{-\mu_k} + t^{-\mu_k} \sum_{m \geq 2} \sum_{\mu_{j_{m,1}} + \mu_{j_{m,2}} + \cdots \mu_{j_{m,m}} = \mu_k} L_m(\xi_{j_{m,1}}, \xi_{j_{m,2}}, \ldots, \xi_{j_{m,m}}) \right\}$$

$$+ \eta_k t^{-\mu_k} + \mu_p \xi_p t^{-(\mu_p + 1)} \right|_{\mu_p + 1 = \mu_k}$$

$$+ O(t^{-\delta}).$$
Thus,

\[ w'_N = -Aw_N + A\xi_{N+1} + O(t^{-\delta}). \]

Lemma

If for some \( \alpha > 0 \),

\[ y' = -Ay + \xi + O(t^{-\alpha}), \]

then

\[ y(t) = A^{-1}\xi + O(t^{-\alpha}). \]

Proof.

\[
y(t) = e^{-tA}y_0 + e^{-tA} \int_0^t e^{\tau A} \xi \, d\tau + \int_0^t e^{-(t-\tau)A} O(\tau^{-\alpha}) \, d\tau
\]

\[
= e^{-tA}y_0 + e^{-tA} A^{-1}(e^{tA}\xi - \xi) + O(t^{-\alpha})
\]

\[
= A^{-1}\xi + O(t^{-\alpha}). \quad \square
\]

Then \( w_N(t) = A^{-1}(A\xi_{N+1}) + O(t^{-\delta}) = \xi_{N+1} + O(t^{-\delta}). \)

Thus,

\[ v_N(t) = \xi_{N+1} t^{-\mu_{N+1}} + O(t^{-(\mu_{N+1}+\delta)}). \]
4. Application to solutions near special periodic orbits
Application (demonstration)

On the plane $n = 2$, $y = (y_1, y_2)$, $r = |y| = \sqrt{y_1^2 + y_2^2}$.

In polar coordinates, i.e., $y(t) = r(t)(\cos(\theta(t)), \sin(\theta(t)))$, assume

$$
\begin{cases}
    r' = (r - 1)(r - 2), \\
    \theta' = 1.
\end{cases}
$$

Then $r = 1, 2$ and $\theta = \theta_0 + t$ are periodic solutions.

The first ($r = 1$) is asymptotically stable, and the second ($r = 2$) is unstable.

Denote the first periodic orbit by $y^*(t) = (\cos(\theta_0 + t), \sin(\theta_0 + t))$. 
Let $z = r - 1$, then

$$z' = z(z - 1) = -z + z^2, \quad z(0) = z_0 \in (-1, 1).$$

Then $z(t)$ admits an expansion:

$$z(t) = \sum_{k=1}^{\infty} q_k(t)e^{-kt},$$

where real-valued polynomials $q_k$’s solve

$$\frac{dq_k}{dt} = (k - 1)q_k + \sum_{j+\ell=k} q_j q_\ell.$$ 

Hence the solution $y(t)$ has expansion

$$y(t) \exp. \sim y^*(t)\left(1 + \sum_{k=1}^{\infty} q_k(t)e^{-kt}\right).$$
Calculations

- \( q_1(t) = \xi_1 e^{-t} \),
- \( q_2'(t) = q_2(t) + \xi_1^2 \). Hence, \( q_2(t) = -e^t \int_{-\infty}^{t} e^{-\tau} \xi_1^2 d\tau = \xi_1^2 \).
- Claim: \( q_k(t) = c_k \xi_1^k \). Indeed, prove by induction,

\[
q'_k = (k - 1)q_k + s_k \xi_1^k, \quad s_k = \sum_{j=1}^{k-1} c_j.
\]

Then, \( q_k(t) = -e^{(k-1)t} \int_{-\infty}^{t} e^{-(k-1)\tau} s_k \xi_1^k d\tau = \frac{s_k}{k-1} \xi_1^k \), where

\[
c_1 = 1, \quad c_k = \frac{1}{k-1} \left( \sum_{j=1}^{k-1} c_j \right) = 1.
\]

Thus, \( y(t) \exp \sim y^*(t) \left( 1 + \sum_{k=1}^{\infty} \xi_1^k e^{-kt} \right) \).

- Explicitly, \( z(t) = \frac{1}{1-(z_0-1)/z_0 e^t} = \frac{\xi_1 e^{-t}}{1-\xi_1 e^{-t}} = \sum_{k=1}^{\infty} \xi_1^k e^{-kt} \), with \( \xi_1 = z_0 / (z_0 - 1) \).
Non-autonomous case I: Exponential perturbation

\[ r' = (r - 1)(r - 2) + \sum_{k=1}^{\infty} p_k(t) e^{-kt}, \quad \theta' = 1. \]

Then

\[ z' = -z + z^2 + \sum_{k=1}^{\infty} p_k(t) e^{-kt}. \]

Similarly,

\[ y(t) \overset{\text{exp.}}{\sim} y^*(t) \left( 1 + \sum_{k=1}^{\infty} q_k(t) e^{-kt} \right), \]

where

\[ \frac{dq_k}{dt} = (k - 1)q_k + \sum_{j+\ell=k} q_j q_\ell + p_k. \]
Non-autonomous case II: Power perturbation

Assume there are \( d_k \in \mathbb{R} \):

\[
r' = (r - 1)(r - 2) + \sum_{k=1}^{\infty} d_k t^{-k}, \quad \theta' = 1.
\]

Then

\[
z' = -z + z^2 + \sum_{k=1}^{\infty} d_k t^{-k}.
\]

We obtain

\[
y(t) \overset{\text{pow.}}{\sim} y^*(t) \left(1 + \sum_{k=1}^{\infty} a_k t^{-k}\right),
\]

where

\[
a_1 = d_1, \quad a_k = (k - 1)a_{k-1} + \sum_{j+\ell=k} a_j a_{\ell} + d_k \quad \text{for} \quad k \geq 2.
\]
THANK YOU FOR YOUR ATTENTION.