Generalized Forchheimer Equations for Porous Media. Part V.

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Project on Generalized Forchheimer Flows: also with Eugenio Aulisa (TTU), Lidia Bloshanskaya (TTU), Tuoc Phan (Tennessee)

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Outline

Introduction

Fluid flows in porous media with velocity u and pressure p:

- Darcy's Law:

$$\alpha u = -\nabla p,$$

- the "two term" law

$$\alpha u + \beta |u|\, u = -\nabla p,$$

- the "three term" law

$$\mathcal{A} u + \mathcal{B}\, |u|\, u + \mathcal{C} |u|^2\, u = -\nabla p.$$

- the "power" law

$$a u + c^n |u|^{n-1}\, u = -\nabla p,$$

Here $\alpha, \beta, a, c, n, \mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ are empirical positive constants.

General Forchheimer equations

Generalizing the above equations as follows

$$g(|u|)u = -\nabla p.$$

Let $G(s) = sg(s)$. Then $G(|u|) = |\nabla p| \Rightarrow |u| = G^{-1}(|\nabla p|)$. Hence

$$u = -\frac{\nabla p}{g(G^{-1}(|\nabla p|))} \Rightarrow u = -K(|\nabla p|)\nabla p,$$

where

$$K(\xi) = K_g(\xi) = \frac{1}{g(s)} = \frac{1}{g(G^{-1}(\xi))}, \quad sg(s) = \xi.$$

We derived a non-linear equation of Darcy type from Forchheimer equations.

IBVP for pressure

Let ρ be the density. Continuity equation

$$\frac{d\rho}{dt} + \nabla \cdot (\rho u) = 0.$$

For slightly compressible fluid:

$$\frac{d\rho}{dp} = \frac{1}{\kappa}\rho,$$

where $\kappa \gg 1$. Then

$$\frac{dp}{dt} = \kappa \nabla \cdot \Big(K(|\nabla|)\nabla p \Big) + K(|\nabla p|)|\nabla p|^2.$$

Since $\kappa \gg 1$, we neglect the last terms, after scaling the time variable:

$$\frac{dp}{dt} = \nabla \cdot \Big(K(|\nabla|)\nabla p \Big).$$

Consider the equation on a bounded domain U in $\mathbb{R}^3$ with boundary Γ.
Dirichlet boundary data:

$$u = \psi(x,t) \text{ is known on } \Gamma.$$

Historical remarks

- Darcy-Dupuit: 1865
- Forchheimer: 1901
- Other nonlinear models: 1940s–1960s
- Incompressible fluids: Payne, Straughan and collaborators since 1990's, Celebi-Kalantarov-Ugurlu since 2005 (Brinkman-Forchheimer)
- Derivation of non-Darcy, non-Forchheimer flows: Marusic-Paloka and Mikelic 2009 (homogenization for Navier–Stokes equations), Balhoff et. al. 2009 (computational)

Works on generalized Forchheimer flows

A. Single-phase flows.

- 1990's Numerical study
- L^2-theory (for slightly compressible flows): Aulisa-Bloshanskaya-Ibragimov-LH (2009), Ibragimov-LH: Dirichlet B.C. (2011), Ibragimov-LH Flux B.C. (2012), Aulisa-Bloshanskaya-Ibragimov total flux, productivity index (2011, 2012).
- L^α-theory : Ibragimiv-Kieu-Sobol-LH (this talk).
- L^∞-theory : Kieu-Phan-LH (in progress).

B. Multi-phase flows.

- One-dimensional case: Ibragimov-Kieu-LH (2013).
- Multi-dimensional case: Ibragimov-Kieu-LH (in preparation).

Note: there are more works on Forchheimer flows (2-terms or 3 terms).

Class $FP(N, \vec{\alpha})$

We introduce a class of "Forchheimer polynomials"

Definition

A function $g(s)$ is said to be of class $FP(N, \vec{\alpha})$ if

$$g(s) = a_0 s^{\alpha_0} + a_1 s^{\alpha_1} + a_2 s^{\alpha_2} + \ldots + a_N s^{\alpha_N} = \sum_{j=0}^{N} a_j s^{\alpha_j},$$

where $N > 0$, $0 = \alpha_0 < \alpha_1 < \alpha_2 < \ldots < \alpha_N$, and $a_0, a_N > 0$, $a_1, \ldots, a_{N-1} \geq 0$.

Notation: $\alpha_N = \deg(g)$, $\vec{a} = (a_0, a_1, \ldots, a_N)$,

$$a = \frac{\alpha_N}{\alpha_N + 1} \in (0, 1), \quad b = \frac{a}{2 - a} = \frac{\alpha_N}{\alpha_N + 2} \in (0, 1).$$

Lemma

Let $g(s,\vec{a})$ be in class FP($N,\vec{\alpha}$). One has for any $\xi \geq 0$ that

$$\frac{C_1(\vec{a})}{(1+\xi)^a} \leq K(\xi,\vec{a}) \leq \frac{C_2(\vec{a})}{(1+\xi)^a},$$

$$C_3(\vec{a})(\xi^{2-a}-1) \leq K(\xi,\vec{a})\xi^2 \leq C_2(\vec{a})\xi^{2-a}.$$

Degree Condition (DC)

$$\deg(g) \leq \frac{4}{n-2} \iff 2 \leq (2-a)^* = \frac{n(2-a)}{n-(2-a)}.$$

Under the (DC), the Sobolev space $W^{1,2-a}(U)$ is continuously embedded into $L^2(U)$.

(NDC)

$$\deg(g) > \frac{4}{n-2} \iff 2 > (2-a)^* = \frac{n(2-a)}{n-(2-a)}.$$

Definitions and notations

Let $\bar{p}(x,t) = p(x,t) - \Psi(x,t)$ where $\Psi(x,t)$ is an extension of $\psi(x,t)$ from ∂U to U.

$$G_1(t) = \int_U |\nabla\Psi(x,t)|^2 dx + \left[\int_U |\Psi_t(x,t)|^{r_0} dx\right]^{\frac{2-a}{r_0(1-a)}} + \left[\int_U |\Psi_t(x,t)|^{r_0} dx\right]^{\frac{1}{r_0}}$$

$$G_2(t) = \int_U |\nabla\Psi_t(x,t)|^2 dx + \int_U |\Psi_t(x,t)|^2 dx,$$

$$G_3(t) = G_1(t) + G_2(t),\ G_4(t) = G_3(t) + \int_U |\Psi_{tt}|^2 dx,$$

where $r_0 = \frac{n(2-a)}{(2-a)(n+1)-n}$. For any $\alpha > 0$, we define

$$\alpha_* = \frac{na}{2-a} = \frac{n\alpha_N}{\alpha_N + 2},$$

$$\widehat{\alpha} = \max\{\alpha, 2, \alpha_*\} = \begin{cases} \max\{\alpha, \alpha_*\} & \text{in the NDC case} \\ \max\{\alpha, 2\} & \text{in the DC case.} \end{cases}$$

Estimates of solutions - I: for p

For $\alpha > 0$ and $t \geq 0$, we define

$$A(\alpha, t) = \int_U |\nabla\Psi|^{\frac{\alpha(2-a)}{2}} dx + \int_U |\Psi_t|^\alpha dx, \quad t \geq 0,$$

$$B(\alpha, t) = e^{-c_5(\alpha)t} \int_U |\bar{p}(x, 0)|^\alpha dx + \int_0^t e^{-c_5(\alpha)(t-\tau)} A(\alpha, \tau) d\tau.$$

Theorem

Let $\alpha > 0$. *Then*

$$\int_U |\bar{p}(x, t)|^{\alpha} dx \leq C(1 + B(\widehat{\alpha}, t))$$

for all $t \geq 0$, *and*

$$\limsup_{t\to\infty} \int_U |\bar{p}|^{\alpha} dx \leq C(1 + \limsup_{t\to\infty} A(\widehat{\alpha}, t)).$$

Ideas of proof

Let $\gamma_0 = \frac{\alpha}{\alpha - a}$. Multiplying equation of $\bar{p}$ by $|\bar{p}|^{\alpha-1}\mathrm{sgn}(\bar{p})$, integrating over U, and many other calculations give

$$\frac{d}{dt}\int_U |\bar{p}|^\alpha dx \le -C\int_U |\nabla \bar{p}|^{2-a}|\bar{p}|^{\alpha-2}dx + \varepsilon\Big(\int_U |\bar{p}|^\alpha dx\Big)^{\frac{1}{\gamma_0}} + C_\varepsilon(1+A(\alpha,t)).$$

Poincaré-Sobolev inequality: for $\alpha \ge \max\{2, \alpha_*\}$, applying the imbedding from the Sobolev space $\mathring{W}^{1,2-a}(U)$ into $L^{\frac{\alpha(2-a)}{\alpha-a}}(U)$ to function $|u|^{\frac{\alpha-a}{2-a}}$ gives

$$\int_U |\nabla u|^{2-a}|u|^{\alpha-2}dx \ge c_2\Big(\int_U |u|^\alpha dx\Big)^{\frac{1}{\gamma_0}}.$$

Using this,

$$\frac{d}{dt}\int_U |\bar{p}|^\alpha dx \le -C\Big(\int_U |\bar{p}|^\alpha dx\Big)^{\frac{1}{\gamma_0}} + \varepsilon\Big(\int_U |\bar{p}|^\alpha dx\Big)^{\frac{1}{\gamma_0}} + C_\varepsilon(1+A(\alpha,t)).$$

Selecting ε sufficiently small, we obtain

$$\frac{d}{dt}\int_U |\bar{p}|^\alpha dx \le -C\Big(\int_U |\bar{p}|^\alpha dx\Big)^{\frac{1}{\gamma_0}} + C(A(\alpha,t)+1).$$

Differential inequality

Then we use the following estimates.

Lemma

Let $\phi : [0,\infty) \to [0,\infty)$ be continuous, strictly increasing. Suppose $y(t) \ge 0$ is a continuous function on $[0,\infty)$ such that

$$y' \le -h(t)\phi^{-1}(y(t)) + f(t), \quad t > 0,$$

where $h(t) > 0$, $f(t) \ge 0$ for $t \ge 0$ are continuous functions on $[0,\infty)$.
(i) If $M(t) = Env\left(\frac{f(t)}{h(t)}\right)$ on $[0,\infty)$ then

$$y(t) \le y(0) + \phi(M(t)) \textit{ for all } t \ge 0.$$

(ii) If $\int_0^\infty h(\tau)d\tau = \infty$ then

$$\limsup_{t\to\infty} y(t) \le \phi\left(\limsup_{t\to\infty} \frac{f(t)}{h(t)}\right).$$

Estimates of solutions - II: for ∇p

Improving L^2-theory. Previously,

$$\frac{d}{dt}\int_U H(|\nabla p|)dx \le -d_5 \int_U H(|\nabla p|)dx + C\int_U \bar{p}^2 dx + CG_3(t),$$

where $H(\nabla p) \approx K(|\nabla p|)|\nabla p|^2 \approx |\nabla p|^{2-a} \pm 1$.

Theorem

If $t \ge 0$ then

$$\int_U |\nabla p(x,t)|^{2-a}dx \le C + Ce^{-d_5 t}\int_U |\bar{p}(x,0)|^{\widehat{2}} + |\nabla p(x,0)|^{2-a}dx$$
$$+ C\int_0^t e^{-d_5(t-\tau)}(A(\widehat{2},\tau) + G_3(\tau))d\tau,$$

where $d_5 > 0$. Consequently,

$$\limsup_{t\to\infty}\int_U |\nabla p(x,t)|^{2-a}dx \le C\limsup_{t\to\infty}(1 + A(\widehat{2},t) + G_3(t)).$$

Estimates of solutions - III: for p_t

Theorem

If $t \ge t_0 > 0$ then

$$\int_U |\nabla p(x,t)|^{2-a} + |\bar{p}_t(x,t)|^2 dx \le C + Ce^{-d_7 t} \int_U |\bar{p}(x,0)|^{\widehat{2}} dx$$
$$+ Ce^{Ct_0} t_0^{-1} e^{-d_7 t} \Big\{ \int_U |\bar{p}(x,0)|^2 + |\nabla p(x,0)|^{2-a} dx + \int_0^{t_0} G_3(\tau) d\tau \Big\}$$
$$+ C \int_0^t e^{-d_7(t-\tau)} (A(\widehat{2},\tau) + G_4(\tau)) d\tau,$$

where $d_7 > 0$. Consequently,

$$\limsup_{t\to\infty} \int_U |\nabla p(x,t)|^{2-a} + |\bar{p}_t(x,t)|^2 dx \le C \limsup_{t\to\infty} (1 + A(\widehat{2},t) + G_4(t)).$$

Estimates of solutions - IV: for mixed terms

$$\frac{d}{dt}\int_U |\bar p|^\alpha dx \le -C\int_U |\nabla \bar p|^{2-a}|\bar p|^{\alpha-2}dx + \varepsilon\Big(\int_U |\bar p|^\alpha dx\Big)^{\frac{1}{\gamma_0}} + C_\varepsilon(1+A(\alpha,t)).$$

Integrate in time or $\int_U |\nabla \bar p|^{2-a}|\bar p|^{\alpha-2}dx \le \alpha\int_U |\bar p_t||\bar p|^{\alpha-1}dx + \dots$

Corollary

If $\alpha \ge \max\{2,\alpha_*\}$ *and* $t>0$ *then*

$$\int_0^t\int_U |\nabla p(x,\tau)|^{2-a}|\bar p(x,\tau)|^{\alpha-2}dx\,d\tau$$
$$\le C\Big(\int_U |\bar p(x,0)|^\alpha dx + \int_0^t (1+A(\alpha,\tau))d\tau\Big),$$
$$\int_U |\nabla p(x,t)|^{2-a}|\bar p(x,t)|^{\alpha-2}dx$$
$$\le C\Big(1+A(\alpha,t)+\int_U |\bar p(x,t)|^{2(\alpha-1)}dx + \int_U |\bar p_t(x,t)|^2 dx\Big).$$

Dependence on the boundary data

- Let $N \geq 1$, the exponent vector $\vec{\alpha}$ and coefficient vector $\vec{a}$ be fixed.
- Let $p_k = p_k(x,t)$ (for $i = 1,2$) be the solution with initial data $p_k(x,0)$ and boundary data ψ_k.
- Let Ψ_1, Ψ_2 be extensions of ψ_1, ψ_2, and $\Phi = \Psi_1 - \Psi_2$.
- Let $\bar{p}_k = p_k - \Psi_k$ for $k = 1,2$, and $z = p_1 - p_2$, $\bar{z} = \bar{p}_1 - \bar{p}_2 = z - \Phi$.

Then

$$\begin{cases} \frac{\partial \bar{z}}{\partial t} = \nabla \cdot (K(|\nabla p_1|)\nabla p_1) - \nabla \cdot (K(|\nabla p_2|)\nabla p_2) + \Phi_t \text{ in } U \times (0,\infty), \\ \bar{z}(x,t) = 0 \text{ on } \Gamma \times (0,\infty). \end{cases}$$

Goal. Estimating $\bar{z}(x,t)$ in terms of $\bar{z}(x,0)$ and Φ.

Main differential inequalities for $\bar{z}$

Define $N(\lambda, t) = \sum_{k=1}^{2} \left(\int_U |\bar{p}_k(x,t)|^{\lambda} dx \right)^{1/\lambda}$ for $\lambda > 0$, and $N(0,t) = 1$.

Lemma

Let $\alpha \geq 2$. (i) In the DC case, for $t > 0$,

$$\frac{d}{dt} \int_U |\bar{z}(x,t)|^{\alpha} dx \leq -c_6 \Big[\int_U |\bar{z}(x,t)|^{\alpha} dx \Big] M_1(t)^{-\frac{a}{2-a}} + CF(\alpha, t) D(\alpha, t)$$

where

$$M_1(t) = 1 + \int_U |\nabla p_1(x,t)|^{2-a} + |\nabla p_2(x,t)|^{2-a} dx,$$

$$F(\alpha, t) = 1 + N(\alpha, t)^{\alpha-1} + N(\gamma_1, t)^{\alpha-2} M_1(t)^{\frac{1-a}{2-a}},$$

with $\gamma_1 = \gamma_1(\alpha) \stackrel{\text{def}}{=} 2(\alpha - 2)(2-a)$, and

$$D(\alpha, t) = \begin{cases} \|\nabla \Phi(\cdot,t)\|_{L^{2(2-a)}} + \|\nabla \Phi(\cdot,t)\|_{L^{\alpha}}^2 + \|\Phi_t(\cdot,t)\|_{L^{\alpha}} & \text{if } \alpha > 2, \\ \|\nabla \Phi(\cdot,t)\|_{L^{2-a}} + \|\nabla \Phi(\cdot,t)\|_{L^2}^2 + \|\Phi_t(\cdot,t)\|_{L^2} & \text{if } \alpha = 2. \end{cases}$$

Lemma (cont.)

(ii) In the NDC case,

$$\frac{d}{dt}\int_U |\bar z(x,t)|^\alpha dx \le -c_7\Big[\int_U |\bar z(x,t)|^\alpha dx\Big]^\theta M_2(\alpha,t)^{-\frac{2-\theta_1}{\theta_1}} + CF(\alpha,t)D(\alpha,t)$$

for all $t>0$, where

$$M_2(\alpha,t) = 1 + \int_U |\nabla p_1(x,t)|^{2-a} + |\nabla p_2(x,t)|^{2-a} + |\bar p_1(x,t)|^{\theta_2\alpha} + |\bar p_2(x,t)|^{\theta_2\alpha} dx.$$

Monotonicity

Multiplying the equation of $\bar z$ by $|\bar z|^{\alpha-1}\mathrm{sgn}(\bar z)$, ...

$$\frac{1}{\alpha}\frac{d}{dt}\int_U |\bar z|^\alpha dx$$
$$= -(\alpha-1)\int_U (K(|\nabla p_1|)\nabla p_1 - K(|\nabla p_2|)\nabla p_2)\cdot(\nabla p_1 - \nabla p_2)|\bar z|^{\alpha-2}dx + \ldots$$

Notation. $\vee$ for max.

Lemma (Degenerate monotonicity)

For any $y, y' \in \mathbb{R}^n$, one has

$$\big(K(|y|)y - K(|y'|)y'\big)\cdot(y-y') \geq (1-a)K(|y|\vee|y'|)|y-y'|^2.$$

$$\frac{d}{dt}\int_U |\bar z|^\alpha dx \leq -C\int_U K(|\nabla p_1|\vee|\nabla p_2|)|\nabla \bar z|^2|\bar z|^{\alpha-2}dx + \ldots$$

Weighted Poincaré–Sobolev inequality

Lemma

Let $\xi(x) \geq 0$ on U and $u(x) = 0$ on ∂U. Assume $\alpha \geq 2$.
(i) In the DC case,

$$\int_U |u|^\alpha dx \leq c_3 \Big[\int_U K(\xi)|\nabla u|^2 |u|^{\alpha-2} dx \Big] \Big[1 + \int_U \xi^{2-a} dx \Big]^{\frac{a}{2-a}}.$$

(ii) In the NDC case, given two numbers θ and θ_1 that satisfy

$$\theta > \frac{2}{(2-a)^*} \text{ and } \max\Big\{1, \frac{2n}{n\theta+2}\Big\} \leq \theta_1 < 2-a,$$

then

$$\int_U |u|^\alpha dx \leq c_4 \Big[\int_U K(\xi)|\nabla u|^2 |u|^{\alpha-2} dx \Big]^{\frac{1}{\theta}} \Big[1 + \int_U \xi^{2-a} + |u|^{\theta_2\alpha} dx \Big]^{\frac{2-\theta_1}{\theta\theta_1}},$$

where $\theta_2 = \frac{\theta_1(\theta-1)(2-a)}{2(2-a-\theta_1)} > 0$.

Dependence - Ia. DC case.

Theorem

(i) Assume (DC) and $\alpha \geq 2$. Then

$$\int_U |\bar{z}(x,t)|^\alpha dx \leq e^{-c_8 \int_0^t \overline{M}_1(\tau)^{\frac{-a}{2-a}} d\tau} \int_U |\bar{z}(x,0)|^\alpha dx + C \int_0^t e^{-c_8 \int_\tau^t \overline{M}_1(s)^{\frac{-a}{2-a}} ds} \overline{F}(\alpha,\tau) D(\alpha,\tau) d\tau$$

for all $t \geq 0$, where $c_8 = c_8(\alpha) > 0$. Moreover, if $\int_0^\infty \tilde{M}_1(t)^{-\frac{a}{2-a}} dt = \infty$ then

$$\limsup_{t\to\infty} \int_U |\bar{z}(x,t)|^\alpha dx \leq C \limsup_{t\to\infty} \Big[\tilde{F}(\alpha,t) \tilde{M}_1(t)^{\frac{a}{2-a}} D(\alpha,t) \Big].$$

Dependence - Ib. NDC case.

Theorem (cont.)

(ii) Assume (NDC) and $\alpha \geq \alpha_*$. *Then*

$$\int_U |\bar{z}(x,t)|^\alpha dx \leq \int_U |\bar{z}(x,0)|^\alpha dx + C\Big[Env\Big(\overline{F}(\alpha,t)\overline{M}_2(\widehat{\theta_2\alpha},t)^{\frac{2-\theta_1}{\theta_1}} D(\alpha,t)\Big)\Big]^{\frac{1}{\theta}}$$

for all $t \geq 0$. *Moreover, if* $\int_0^\infty \tilde{M}_2(t)^{-\frac{2-\theta_1}{\theta_1}} dt = \infty$ *then*

$$\limsup_{t\to\infty} \int_U |\bar{z}(x,t)|^\alpha dx \leq C \limsup_{t\to\infty} \Big[\tilde{F}(\alpha,t)\tilde{M}_2(\widehat{\theta_2\alpha},t)^{\frac{2-\theta_1}{\theta_1}} D(\alpha,t)\Big]^{\frac{1}{\theta}}.$$

Dependence - Ic

Corollary

Let $\alpha \geq \max\{2, \alpha_*\}$. *Assume the functions* Ψ_k $(k = 1, 2)$ *satisfy*

$$\sup_{[0,\infty)} \{\|\nabla\Psi_k(\cdot, t)\|_{L^\infty}, \|(\Psi_k)_t(\cdot, t)\|_{L^\infty}, \|\nabla(\Psi_k)_t(\cdot, t)\|_{L^\infty}\} < \infty,$$

$$\lim_{t\to\infty} \|\nabla\Phi(\cdot, t)\|_{L^\gamma} = \lim_{t\to\infty} \|\Phi_t(\cdot, t)\|_{L^\alpha} = 0,$$

where $\gamma = \max\{\alpha, 2(2 - a)\}$. *Then*

$$\lim_{t\to\infty} \int_U |\bar{z}(x, t)|^\alpha dx = 0.$$

Gradient

Lemma

In the NDC case, we have for all $t > 0$ that

$$\Big[\int_U |\nabla z|^{2-a} dx\Big]^{\frac{2}{2-a}} \le C M_1(t)^{\frac{1}{2-a}} \|\nabla \Phi(\cdot,t)\|_{L^{2-a}} + C M_1(t)^{\frac{a}{2-a}} M_3(t)^{\frac12} \Big[\int_U |\bar z|^{\alpha_*} dx\Big]^{\frac{1}{\alpha_*}},$$

where

$$M_3(t) = \int_U |(\bar p_1)_t(x,t)|^2 + |(\bar p_2)_t(x,t)|^2 dx + \int_U |\Phi_t(x,t)|^2 dx.$$

Theorem

Assume (NDC).
(i) Then

$$\Big(\int_U |\nabla z(x,t)|^{2-a}dx\Big)^{\frac{2}{2-a}} \le C\overline{M}_1(t)^{\frac{1}{2-a}}\|\nabla\Phi(\cdot,t)\|_{L^{2-a}} + C\overline{M}_1(t)^{\frac{a}{2-a}}\overline{M}_3(t)^{\frac12}\overline{D}(t)$$

for all $t \ge 1$*, where*

$$\overline{D}(t) = \|\bar z(\cdot,0)\|_{L^{\alpha_*}} + \Big[Env\Big(\overline{F}(\alpha_*,t)\overline{M}_2(\widehat{\theta_2\alpha_*},t)^{\frac{2-\theta_1}{\theta_1}}D(\alpha_*,t)\Big)\Big]^{\frac{1}{\theta\alpha_*}}.$$

Theorem

(ii) Let $\eta_1 = 1 + \bar{A}(\mu_1) + \limsup_{t\to\infty} \mathcal{G}_4(t)$. *If* $\eta_1 < \infty$ *then*

$$\limsup_{t\to\infty} \Big(\int_U |\nabla z(x,t)|^{2-a} dx \Big)^{\frac{2}{2-a}} \le C\eta_1^{\frac{1}{2-a}} \limsup_{t\to\infty} \|\nabla\Phi(\cdot,t)\|_{L^{2-a}}$$
$$+ C\eta_1^{\mu_3} \limsup_{t\to\infty} \Big\{ \|\nabla\Phi(\cdot,t)\|_{L^{2(2-a)}} + \|\nabla\Phi(\cdot,t)\|^2_{L^{\alpha_*}} + \|\Phi_t(\cdot,t)\|_{L^{\alpha_*}} \Big\}^{\frac{1}{\theta\alpha_*}}.$$

Case $\eta_1 = \infty$

Assume (NDC) and $\eta_1 = \infty$. Define for $t \geq 0$,

$$\omega(t) = \begin{cases} 1 + \bar{A}(\mu_1) + \int_0^t e^{-d_7(t-\tau)} \mathcal{G}_4(\tau) d\tau & \text{if } \bar{A}(\mu_1) < \infty, \\ 1 + \bar{\beta}(\mu_1)^{\frac{\mu_1}{\mu_1 - 2a}} + \tilde{\mathcal{A}}(\mu_1, t) & \text{if } \bar{A}(\mu_1) = \infty, \\ \quad + \int_0^t e^{-d_7(t-\tau)} (\tilde{\mathcal{A}}(\mu_1, \tau) + \mathcal{G}_4(\tau)) d\tau & \quad \bar{\beta}(\mu_1) < \infty, \\ 1 + Env\tilde{\mathcal{A}}(\mu_1, t) + \int_0^t e^{-d_7(t-\tau)} \mathcal{G}_4(\tau) d\tau & \text{if } \bar{A}(\mu_1) = \infty, \\ & \quad \bar{\beta}(\mu_1) = \infty. \end{cases}$$

Dependence III. Asymptotics (b)

Theorem

Assume (NDC) and $\eta_1 = \infty$. *If* $\int_0^\infty \omega(t)^{-\mu_4}dt = \infty$ *and*

$$\eta_2 \stackrel{\text{def}}{=\!=} \limsup_{t\to\infty}[(\omega(t)^{\mu_5})']^+ < \infty$$

then

$$\limsup_{t\to\infty}\Big(\int_U |\nabla z(x,t)|^{2-a}dx\Big)^{\frac{2}{2-a}} \le C\limsup_{t\to\infty}\Big(\omega(t)^{\frac{1}{2-a}}\|\nabla\Phi(\cdot,t)\|_{L^{2-a}}\Big)$$
$$+ C\Big[\limsup_{t\to\infty}\omega(t)^{\mu_6}D(t)\Big]^{\frac{1}{\theta\alpha_*}} + C\eta_2^{\frac{1}{\theta\alpha_*}}\Big[\limsup_{t\to\infty}\omega(t)^{\mu_7}D(t)\Big]^{\frac{1}{\theta^2\alpha_*}}.$$

Dependence on the Forchheimer polynomial

Collaborator Thinh Kieu will present this topic next week in this Applied Math Seminar.

THANK YOU FOR YOUR ATTENTION!