Non-linear Problems in Fluid Dynamics

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Colloquium
Department of Mathematics and Statistics
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October 1, 2013
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PART I. 3-D Navier-Stokes Equations

IA. Theory of normal forms.

IB. Global regularity in thin domains.
Navier-Stokes equations (NSE) in $\mathbb{R}^3$ with a potential body force

\[
\begin{aligned}
\frac{\partial u}{\partial t} + (u \cdot \nabla) u - \nu \Delta u &= -\nabla p + f, \\
\text{div } u &= 0, \\
\mathbf{u}(x,0) &= \mathbf{u}^0(x),
\end{aligned}
\]

$\nu > 0$ is the kinematic viscosity,  
$\mathbf{u} = (u_1, u_2, u_3)$ is the unknown velocity field,  
$p \in \mathbb{R}$ is the unknown pressure,  
$f$ is the body force,  
$\mathbf{u}^0$ is the initial velocity.
Existence and Uniqueness

Denote by $N$ the outward normal vector to the boundary. Define

$$H = \{ u \in L^2(\Omega) : \nabla \cdot u = 0 \text{ in } \Omega, \ u \cdot N = 0 \text{ on } \partial \Omega \},$$

$$V = \{ u \in H^1(\Omega) : u = 0 \text{ on } \partial \Omega \}.$$ 

[Leray 1933, 1934] Suppose $f = f(x) \in L^2(\Omega)$.

- If $u_0 \in H$, then there exists a weak solution on $[0, \infty)$:
  $$u \in C([0, \infty); H_{\text{weak}}) \cap L^\infty(0, \infty; H) \cap L^2(0, \infty; V).$$

  **Question 1:** Is this weak solution unique?

- If $u_0 \in V$, then there exists a unique strong solution on $[0, T)$ for some $T > 0$:
  $$u \in C([0, \infty); V) \cap L^\infty(0, \infty; V) \cap L^2(0, \infty; H^2(\Omega)).$$

  **Question 2:** Can it be $T = \infty$?

In the 2D case, Questions 1 and 2 have affirmative answers.

In the 3D case, still open!

**Small data results:** If $\|u_0\|_{H^1(\Omega)}$ and $\|f\|_{L^2(\Omega)}$ are small then the strong solution exists for all $t > 0$. 

Part IA. Theory of normal forms

• \( f = -\nabla \phi \) is a potential body force.
• Let \( L > 0 \) and \( \Omega = (0, L)^3 \). The L-periodic solutions:
  \[
  u(x + Le_j) = u(x) \text{ for all } x \in \mathbb{R}^3, j = 1, 2, 3.
  \]
• Zero average condition
  \[
  \int_{\Omega} u(x) \, dx = 0,
  \]
• Throughout \( L = 2\pi \) and \( \nu = 1 \).
• Let \( \mathcal{V} \) be the set of \( \mathbb{R}^3 \)-valued L-periodic trigonometric polynomials which are divergence-free and satisfy the zero average condition. We define
  \[
  H = \text{closure of } \mathcal{V} \text{ in } L^2(\Omega)^3 = H^0(\Omega)^3, \\
  V = \text{closure of } \mathcal{V} \text{ in } H^1(\Omega)^3, \\
  D_A = \text{closure of } \mathcal{V} \text{ in } H^2(\Omega)^3.
  \]
• Norm on \( H \): \( |u| = \|u\|_{L^2(\Omega)} \), on \( V \): \( \|u\| = |\nabla u| \), on \( D_A \): \( |\Delta u| \).
Functional form of NSE

• The Stokes operator:

\[ Au = -\Delta u \text{ for all } u \in D_A. \]

• The bilinear mapping:

\[ B(u, v) = P_L(u \cdot \nabla v) \text{ for all } u, v \in D_A, \]

where \( P_L \) is the Leray projection from \( L^2(\Omega) \) onto \( H \).

• Denote by \( \mathcal{R} \) the set of all initial data \( u^0 \in V \) such that the solution \( u(t) \) is regular for all \( t > 0 \). The functional form of the NSE:

\[ \frac{du(t)}{dt} + Au(t) + B(u(t), u(t)) = 0, \quad t > 0, \]

\[ u(0) = u^0 \in \mathcal{R}, \]

where the equation holds in \( D_A \) for all \( t > 0 \) and \( u(t) \) is continuous from \([0, \infty)\) into \( V \).
Poincaré–Dulac theory for ODE

Consider an ODE in $\mathbb{R}^n$ of in the formal series form:

$$\frac{dx}{dt} + Ax + \Phi^2(x) + \Phi^3(x) + \ldots = 0, \ x \in \mathbb{R}^n,$$

- $A$ is a linear operator from $\mathbb{R}^n$ to $\mathbb{R}^n$
- each $\Phi^d$ is a homogeneous polynomial of degree $d$ from $\mathbb{R}^n$ to $\mathbb{R}^n$

There exists a formal series $y = x + \sum_{d=1}^{\infty} \psi^d(x)$, where $\psi^d$ is a homogeneous polynomial of degree $d$ from $\mathbb{R}^n$ to $\mathbb{R}^n$, which transforms the above ODE into an equation

$$\frac{dy}{dt} + Ay + \Theta^2(y) + \Theta^3(y) + \ldots = 0, \ y \in \mathbb{R}^n,$$

where all monomials of each $\Theta^d$ are resonant.

**Resonance.** Matrix $A$ has eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding eigenvectors $\xi_1, \ldots, \xi_n$. For each $x \in \mathbb{R}^n$, let $x_i$ be its coordinate with respect to $\xi_i$. A monomial $x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_n^{\alpha_n}\xi_k$ is called **resonant** if

$$\lambda_k = \alpha_1\lambda_1 + \alpha_2\lambda_2 + \ldots + \alpha_n\lambda_n.$$
Poincaré–Dulac normal form for NSE

Functional form of NSE:

\[ \frac{du}{dt} + Au + B(u, u) = 0. \]

A differential equation in an infinite dimensional space \( E \)

\[ \frac{d\xi}{dt} + A\xi + \sum_{d=2}^{\infty} \Phi[d](\xi) = 0 \quad (\star) \]

is a Poincaré–Dulac normal form for the NSE if

1. Each \( \Phi[d] \in \mathcal{H}[d](E) \), the space of homogeneous polynomials of order \( d \), and \( \Phi[d](\xi) = \sum_{k=1}^{\infty} \Phi_k[d](\xi) \), where all \( \Phi_k[d] \in \mathcal{H}[d](E) \) are resonant monomials,

2. Equation (\( \star \)) is obtained from NSE by a formal change of variable \( u = \sum_{d=1}^{\infty} \Psi[d](\xi) \) where \( \Psi[d] \in \mathcal{H}[d](E) \).
Asymptotic expansion - Normalization map

For \( u_0 \in \mathcal{R} \), the solution \( u(t) \) has an asymptotic expansion: [Foias-Saut]

\[
u(t) \sim q_1(t)e^{-t} + q_2(t)e^{-2t} + q_3(t)e^{-3t} + \ldots,
\]

where \( q_j(t) = W_j(t, u^0) \) is a polynomial in \( t \) of degree at most \( (j - 1) \) and with values are trigonometric polynomials. This means that for any \( N \in \mathbb{N}, m \in \mathbb{N}, \)

\[
\|u(t) - \sum_{j=1}^{N} q_j(t)e^{-jt}\|_{H^m(\Omega)} = O(e^{-(N+\varepsilon)t})
\]
as \( t \to \infty \), for some \( \varepsilon = \varepsilon_{N,m} > 0 \).

Let \( W(u^0) = \xi_1 \oplus \xi_2 \oplus \cdots \), where \( \xi_j = R_j q_j(0) \), for \( j = 1, 2, 3 \ldots \). Then \( W \) is an one-to-one analytic mapping from \( \mathcal{R} \) to the Fréchet space

\[
S_A = R_1 H \oplus R_2 H \oplus \cdots.
\]
Constructions of polynomials $q_j(t)$

If $u^0 \in \mathcal{R}$ and $W(u^0) = (\xi_1, \xi_2, ...)$, then $q_j$’s are the unique polynomial solutions to the following equations

$$q_j' + (A - j)q_j + \beta_j = 0,$$

with $R_j q_j(0) = \xi_j$, where $\beta_j$’s are defined by

$$\beta_1 = 0 \text{ and for } j > 1, \quad \beta_j = \sum_{k+l=j} B(q_k, q_l).$$

Explicitly, these polynomials $q_j(t)$’s are recurrently given by

$$q_j(t) = \xi_j - \int_0^t R_j \beta_j(\tau) d\tau$$

$$+ \sum_{n \geq 0} (-1)^{n+1} [(A - j)(I - R_j)]^{-n-1} \left( \frac{d}{dt} \right)^n (I - R_j) \beta_j,$$

where $[(A - j)(I - R_j)]^{-n-1} u(x) = \sum_{|k|^2 \neq j} \frac{a_k}{(|k|^2 - j)^{n+1}} e^{ik \cdot x}$, for $u(x) = \sum_{|k|^2 \neq j} a_k e^{ik \cdot x} \in \mathcal{V}$. 

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Normal form

• The $S_A$-valued function $\xi(t) = (\xi_n(t))_{n=1}^{\infty} = (W_n(u(t)))_{n=1}^{\infty} = W(u(t))$ satisfies the following system of differential equations (normal form in $S_A$):

$$\frac{d\xi_1(t)}{dt} + A\xi_1(t) = 0,$$
$$\frac{d\xi_j(t)}{dt} + A\xi_j(t) + \sum_{k+l=j} R_jB(\mathcal{P}_k(\xi(t)), \mathcal{P}_l(\xi(t))) = 0, \quad n > 1,$$

where $P_j(\xi) = q_j(0, \xi)$ for $\xi \in S_A$.

• For $d \geq 1$, let $\mathcal{P}^{[d]}(\xi) = \sum_{j=d}^{\infty} \mathcal{P}^{[d]}_j(\xi) = \sum_{j=d}^{\infty} q_j^{[d]}(0, \xi)$.

For $d \geq 2$, let $B^{[d]}(\xi) = \sum_{j=1}^{\infty} B^{[d]}_j(\xi)$

$$= \sum_{j=1}^{\infty} \sum_{k+l=j} \sum_{m+n=d} R_jB(\mathcal{P}^{[m]}_k(\xi), \mathcal{P}^{[n]}_l(\xi)).$$

Rewrite in the power series form:

$$\frac{d}{dt} \xi + A\xi + \sum_{d=2}^{\infty} B^{[d]}(\xi) = 0.$$
Main result

Let $E^\infty$ be the Fréchet space $C^\infty(\mathbb{R}^3, \mathbb{R}^3) \cap V$.

**Theorem (Foias-H.-Saut 2011)**

The formal power series change of variable

$$u = \xi + \sum_{d=2}^{\infty} P^d(\xi),$$

where $\xi \in E^\infty = C^\infty(\mathbb{R}^3, \mathbb{R}^3) \cap V$, reduces the NSE to a Poincaré–Dulac normal form

$$\frac{d}{dt} \xi + A\xi + \sum_{d=2}^{\infty} B^d(\xi) = 0.$$

- Along the way, $P^d(\xi), B^d(\xi)$ are proved to converge in appropriate Sobolev spaces (depending on $d$).
- The change of variables, in fact, is the formal inverse of the normalization map $W$. 
Explicit change of variable

\[ u \sim \sum_{j=1}^{\infty} q_j(0, \xi) = \sum_{j=1}^{\infty} \sum_{d=1}^{j} q_j^{[d]}(0, \xi) = \sum_{d=1}^{\infty} \sum_{j=d}^{\infty} q_j^{[d]}(0, \xi) = \sum_{d=1}^{\infty} \mathcal{P}^{[d]}(\xi). \]

This has the formal inverse

\[ \xi = \tilde{\mathcal{P}}(u) \overset{\text{def}}{=} u + \sum_{d=2}^{\infty} \tilde{\mathcal{P}}^{[d]}(u) = \sum_{d=1}^{\infty} \tilde{\mathcal{P}}^{[d]}(u). \]

\[
\frac{d}{dt} \xi + A\xi + \sum_{d=2}^{\infty} Q^{[d]}(\xi) = 0.
\]

Explicitly, \( Q^{[1]}(\xi) = A\xi \), and for \( d \geq 2 \),

\[
Q^{[d]}(\xi) = \sum_{k+l=d} B(\mathcal{P}^{[k]}(\xi), \mathcal{P}^{[l]}(\xi)) - \sum_{2 \leq k, l \leq d-1 \overset{k+l=d+1}{\text{and}}} D\mathcal{P}^{[k]}(\xi)(Q^{[l]}(\xi)) + H_A^{(d)} \mathcal{P}^{[d]}(\xi),
\]

where \( H_A^{(d)} \mathcal{P}^{[d]}(\xi) = A\mathcal{P}^{[d]}(\xi) - D\mathcal{P}^{[d]}(\xi)A\xi \) (Poincaré homology oper.).

Proved \( Q^{[d]}(\xi) = \mathcal{B}^{[d]}(\xi) \) for all \( \xi \in E^{\infty} \) and \( d \geq 2 \).
Well-posedness in Banach spaces

**Constructed normed spaces.** Let \((\tilde{\kappa}_n)_{n=2}^{\infty}\) be a fixed sequence of real numbers in the interval \((0, 1]\) satisfying

\[
\lim_{n \to \infty} (\tilde{\kappa}_n)^{1/2^n} = 0.
\]

We define the sequence of positive weights \((\rho_n)_{n=1}^{\infty}\) by

\[
\rho_1 = 1, \quad \rho_n = \tilde{\kappa}_n \gamma_n \rho_{n-1}^2, \quad n > 1,
\]

where \(\gamma_n \in (0, 1]\) are known and decrease to zero faster than \(n^{-n}\).

For \(\bar{u} = (u_n)_{n=1}^{\infty} \in V^{\infty}\), let

\[
\|\bar{u}\|_\star = \sum_{n=1}^{\infty} \rho_n \|
abla u_n\|_{L^2(\Omega)},
\]

Define \(V^\star = \{ \bar{u} \in V^{\infty} : \|\bar{u}\|_\star < \infty \}\), \(S_A^\star = S_A \cap V^\star\). Clearly \(V^\star\) and \(S_A^\star\) are Banach spaces.
Well-posedness of the normal form

We summarize our results in the commutative diagram

\[
\begin{array}{cccccc}
R & \xrightarrow{S(t)} & R \\
\downarrow{W(\cdot)} & & \downarrow{W(\cdot)} \\
S^*_A & \xrightarrow{S_{\text{normal}}(t)} & S^*_A \\
\downarrow{W(0,\cdot)} & & \downarrow{W(0,\cdot)} \\
V^* & \xrightarrow{S_{\text{ext}}(t)} & V^*
\end{array}
\]

Figure: Commutative diagram

where all mappings are continuous.
Part IB. Global regularity in thin domains

Fast Rotation. Chemin, Babin-Mahalov-Nicolaenko, . . .

Thin domains.

- Damped hyperbolic equations in thin domains: Hale-Raugel 1992
- NSE on thin domains: Raugel-Sell 1993, 1994
- Spherical domains: Temam-Ziane 1996

Navier friction boundary conditions.

\[ u \cdot N = 0, \quad \nu [D(u) N]_{tan} + \gamma u = 0, \]

- \( \gamma = \infty \): Dirichlet condition.
- \( \gamma = 0 \): Navier boundary conditions (without friction/free slip).
  - One-layer domain: flat bottom [Iftimie-Raugel-Sell 2005]
  - If the boundary is flat, say, \( x_3 = const \), then
    \[ u_3 = 0, \quad \gamma u_1 + \nu \partial_3 u_1 = \gamma u_2 + \nu \partial_3 u_2 = 0. \] See [Hu 2007].
Single-layer domains

Theorem (H.-Sell 2010, H. 2011)

Consider NSE in $\Omega = \mathbb{T}^2 \times (h_0(\varepsilon), h_1(\varepsilon))$ with Navier boundary conditions on the top and bottom boundaries.
If $\|u_0\|_{\mathcal{V}}, \|f\|_{\mathcal{H}} = o(\varepsilon^{-1/2})$ as $\varepsilon \to 0$, then the strong solution exists for all time when $\varepsilon$ is sufficiently small.

Note: We also prove the existence of the global attractor and estimate its size.
Two-layer domains

\[ \Omega^+_\varepsilon = \{(x_1, x_2, x_3) : (x_1, x_2) \in \mathbb{T}^2, h_0(x_1, x_2) < x_3 < h_+(x_1, x_2)\}, \]

\[ \Omega^-_\varepsilon = \{(x_1, x_2, x_3) : (x_1, x_2) \in \mathbb{T}^2, h_-(x_1, x_2) < x_3 < h_0(x_1, x_2)\}, \]

where \( h_0 = \varepsilon g_0 \), \( h_+ = \varepsilon (g_0 + g_+) \), and \( h_- = \varepsilon (g_0 - g_-) \).

- \( g_+(x_1, x_2), g_+(x_1, x_2) \geq c_0 > 0 \).
- Top \( \Gamma_+ \), bottom \( \Gamma_- \), and interface \( \Gamma_0 \).
- Let \( \Omega_\varepsilon = [\Omega^+_\varepsilon, \Omega^-_\varepsilon], \partial \Omega_\varepsilon = [\partial \Omega^+_\varepsilon, \partial \Omega^-_\varepsilon], \Gamma = [\Gamma_+, \Gamma_-] \).
- Consider the Navier–Stokes equations in \( \Omega_\varepsilon \)

\[
\begin{cases}
\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u = -\nabla p + f, \\
\text{div } u = 0, \\
u(x, 0) = u^0(x),
\end{cases}
\]

where \( u = [u_+, u_-], p = [p_+, p_-], f = [f_+, f_-], \nu = [\nu_+, \nu_-], \)

\( u^0(x) = [u^0_+, u^0_-] \).
Boundary conditions

Let $N_+, N_-$ be the outward normal vector to the boundary of $\Omega_\varepsilon^+, \Omega_\varepsilon^-$. 

- The slip boundary condition on $\partial \Omega_\varepsilon$ is
  \[ u_+ \cdot N_+ = 0 \text{ on } \Gamma_+, \Gamma_0; \quad u_- \cdot N_- = 0 \text{ on } \Gamma_-, \Gamma_0. \]

- The Navier friction conditions on the top and bottom boundaries are
  \[ [\nu_+(Du_+) N_+]_{\text{tan}} + \gamma_+ u_+ = 0 \quad \text{on } \Gamma_+, \]
  \[ [\nu_-(Du_-) N_-]_{\text{tan}} + \gamma_- u_- = 0 \quad \text{on } \Gamma_. \]

- The interface boundary condition on $\Gamma_0$ is
  \[ [\nu_+(Du_+) N_+]_{\text{tan}} + \gamma_0(u_+ - u_-) = 0, \]
  \[ [\nu_-(Du_-) N_-]_{\text{tan}} + \gamma_0(u_- - u_+) = 0. \]

- Assumptions on coefficients: with $2/3 \leq \delta \leq 1$,
  \[ C^{-1}\varepsilon^\delta \leq \gamma_-, \gamma_+ \leq C\varepsilon^\delta, \quad C^{-1}\varepsilon \leq \gamma_0 \leq C\varepsilon. \]
Main results

Stokes operator $A$. Some appropriate averaging operator: $\overline{M}$.

**Theorem (H. 2013)**

There are positive numbers $\varepsilon_\ast$ and $\kappa$ such that if $\varepsilon < \varepsilon_\ast$ and $u_0 \in V$, $f \in L^\infty(0, \infty; L^2(\Omega_\varepsilon)^3)$ satisfy that

$$
\| \overline{M}u_0 \|_{L^2}^2, \quad \varepsilon \| A^{1/2} u_0 \|_{L^2}^2, \quad \varepsilon^{1-\delta} \| \overline{M}f \|_{L^\infty L^2}^2, \quad \varepsilon \| f \|_{L^\infty L^2}^2 \leq \kappa,
$$

then the strong solution $u(t)$ of NSE exists for all $t \geq 0$. Moreover,

$$
\| u(t) \|_{L^2}^2 \leq C \kappa, \quad \| A^{1/2} u(t) \|_{L^2}^2 \leq C \varepsilon^{-1} \kappa, \quad t \geq 0,
$$

where $C > 0$ is independent of $\varepsilon, u_0, f$, and

$$
\| A^{1/2} u \|_{L^2}^2 = 2 \int_\Omega \nu |Du|^2 dx + 2 \int_\Gamma \gamma |u|^2 d\sigma + 2 \gamma_0 \int_{\Gamma_0} |u_+ - u_-|^2 d\sigma.
$$

Remark: The condition on $u_0$ is acceptable.
**Key estimates**

**Proposition**

*If $\varepsilon > 0$ is sufficiently small, then for all $u \in D_A$,*

\[
\|Au + \Delta u\|_{L^2} \leq C\varepsilon\|\nabla^2 u\|_{L^2} + C\|\nabla u\|_{L^2} + C\varepsilon^{\delta-1}\|u\|_{L^2},
\]

\[
C\|Au\|_{L^2} \leq \|u\|_{H^2} \leq C'\|Au\|_{L^2}.
\]

**Proposition**

*Given $\alpha > 0$, there is $C_\alpha > 0$ such that for any $\varepsilon \in (0, 1]$ and $u \in D_A$, we have*

\[
|\langle u \cdot \nabla u, Au \rangle| \leq \alpha\|u\|_{H^2}^2 + C\varepsilon^{1/2}\|A^{1/2}u\|_{L^2}\|Au\|_{L^2}^2
\]

\[
+ C_\alpha \left(\|u\|_{L^2}^2 + [\varepsilon\|A^{1/2}u\|_{L^2}^2\|u\|_{L^2}^2]^{1/3}\right) (\varepsilon^{-1}\|A^{1/2}u\|_{L^2}^2).
\]

*where $C > 0$ is independent of $\varepsilon$ and $\alpha$.***
PART II. Forchheimer flows in porous media

IIA. Single-phase slightly compressible fluids.
IIB. Two-phase incompressible fluids.
Fluid flows in porous media with velocity $u$ and pressure $p$:

- **Darcy’s Law:**
  \[ \alpha u = -\nabla p, \]

- the Forchheimer “two term” law
  \[ \alpha u + \beta |u| u = -\nabla p, \]

- the Forchheimer “three term” law
  \[ Au + B|u| u + C|u|^2 u = -\nabla p. \]

- the Forchheimer “power” law
  \[ au + c^n|u|^{n-1} u = -\nabla p, \]

Here $\alpha$, $\beta$, $a$, $c$, $n$, $A$, $B$, and $C$ are empirical positive constants.
Generalized Forchheimer equations

\[ g(|u|)u = -\nabla p. \]

Let \( G(s) = sg(s) \). Then \( G(|u|) = |\nabla p| \Rightarrow |u| = G^{-1}(|\nabla p|) \). Hence

\[ u = -\frac{\nabla p}{g(G^{-1}(|\nabla p|))} \Rightarrow u = -K(|\nabla p|)\nabla p, \]

\[ K(\xi) = K_g(\xi) = \frac{1}{g(\xi)} = \frac{1}{g(G^{-1}(\xi))}, \quad sg(s) = \xi. \]

Class \( \text{FP}(N, \bar{\alpha}) \). Let \( N > 0, 0 = \alpha_0 < \alpha_1 < \alpha_2 < \ldots < \alpha_N \),

\[ \text{FP}(N, \bar{\alpha}) = \left\{ g(s) = a_0 s^{\alpha_0} + a_1 s^{\alpha_1} + a_2 s^{\alpha_2} + \ldots + a_N s^{\alpha_N} \right\}, \]

where \( a_0, a_N > 0, a_1, \ldots, a_{N-1} \geq 0 \). Notation: \( \alpha_N = \text{deg}(g) \),

\( \bar{\alpha} = (a_0, a_1, \ldots, a_N), a = \frac{\alpha_N}{\alpha_{N+1}} \in (0,1), b = \frac{\alpha_N}{\alpha_{N+2}} \in (0,1). \)
Historical remarks

- Darcy-Dupuit: 1865
- Forchheimer: 1901
- Other nonlinear models: 1940s–1960s
- Incompressible fluids: Payne, Straughan and collaborators since 1990’s, Celebi-Kalantarov-Ugurlu since 2005 (Brinkman-Forchheimer)
- Derivation of non-Darcy, non-Forchheimer flows: Marusic-Paloka and Mikelic 2009 (homogenization for Navier–Stokes equations), Balhoff et. al. 2009 (computational)
Works on generalized Forchheimer flows

A. Single-phase flows.

- 1990’s Numerical study
- $L^\alpha$-theory: H.-Ibragimov-Kieu-Sobol (2012-preprint)

B. Multi-phase flows.


Note: there are more works on Forchheimer flows (2-terms or 3 terms).
Let $\rho$ be the density. Continuity equation

$$\frac{d\rho}{dt} + \nabla \cdot (\rho u) = 0.$$ 

For slightly compressible fluid:

$$\frac{d\rho}{dp} = \frac{1}{\kappa} \rho,$$

where $\kappa \gg 1$. Then

$$\frac{dp}{dt} = \kappa \nabla \cdot \left( K(|\nabla|) \nabla p \right) + K(|\nabla p|) |\nabla p|^2.$$ 

Since $\kappa \gg 1$, we neglect the last terms, after scaling the time variable:

$$\frac{dp}{dt} = \nabla \cdot \left( K(|\nabla|) \nabla p \right).$$

Flux condition on the boundary:

$$-K(|\nabla p|) \nabla p \cdot \tilde{v} = u \cdot \tilde{v} = \psi(x, t), \quad x \in \Gamma, \quad t > 0,$$

where $\tilde{v}$ is the outward normal vector on $\Gamma$ and the flux $\psi(x, t)$ is known.
Lemma

Let \( g(s, \bar{a}) \) be in class \( FP(N, \bar{\alpha}) \). One has for any \( \xi \geq 0 \) that

\[
\frac{C_1(\bar{a})}{(1 + \xi)^a} \leq K(\xi, \bar{a}) \leq \frac{C_2(\bar{a})}{(1 + \xi)^a},
\]

\[
C_3(\bar{a})(\xi^{2-a} - 1) \leq K(\xi, \bar{a})\xi^2 \leq C_2(\bar{a})\xi^{2-a}.
\]

Degree Condition (DC)

\[
\deg(g) \leq \frac{4}{n-2} \iff 2 \leq (2 - a)^* = \frac{n(2-a)}{n-(2-a)}.
\]

Under the (DC), the Sobolev space \( W^{1,2-a}(U) \hookrightarrow L^2(U) \).

Strict Degree Condition (SDC)

\[
\deg(g) < \frac{4}{n-2} \iff 2 < (2 - a)^* = \frac{n(2-a)}{n-(2-a)}.
\]
The initial data

\[ p(x, 0) = p_0(x) \text{ is given.} \]

Let \( \bar{p}(x, t) = p(x, t) - \frac{1}{|U|} \int_U p(x, t) \, dx \). Then

\[ \int_U \bar{p}(x, t) \, dx = 0 \quad \text{for all } t \geq 0, \]

and

\[ \bar{p}(x, t) = p(x, t) - \frac{1}{|U|} \int_U p_0(x) \, dx + \frac{1}{|U|} \int_0^t \int_{\Gamma} \psi(x, \tau) \, d\sigma \, d\tau. \]
$L^2$-estimates

Let $f(t) = \|\psi(t)\|_{L^\infty}^{2} + \|\psi(t)\|_{L^\infty}^{1-a}$ and $\tilde{f}(t) = \|\psi_t(t)\|_{L^\infty}^{2} + \|\psi_t(t)\|_{L^\infty}^{1-a}$. Let $M_f(t)$ be a continuous, increasing majorant of $f(t)$ on $[0, \infty)$. Let $A = \limsup_{t \to \infty} f(t)$ and $\beta = \limsup_{t \to \infty} [f'(t)]^-$. 

**Theorem (Ibragimov 2012)**

Assume $(DC)$, then

$$\|\bar{p}(t)\|_{L^2}^{2} \leq \|\bar{p}_0\|_{L^2}^{2} + C(1 + M_f(t)^{\frac{2}{2-a}}) \text{ for all } t \geq 0.$$ 

Moreover, if $A < \infty$ then

$$\limsup_{t \to \infty} \|\bar{p}(t)\|_{L^2}^{2} \leq C(A + A^{\frac{2}{2-a}}),$$

and if $\beta < \infty$ then there is $T > 0$ such that

$$\|\bar{p}(t)\|_{L^2}^{2} \leq C\left(1 + \beta^{\frac{1}{1-a}} + f(t)^{\frac{2}{2-a}}\right) \text{ for all } t > T.$$
We use the following notation:

\[ m_1(t) = 1 + \|\bar{p}_0\|_{L^2}^2 + M_f(t)^{\frac{2}{2-a}} + \int_{t-1}^{t} \tilde{f}(\tau) d\tau, \]

\[ m_2(t) = 1 + A^{\frac{2}{2-a}} + \int_{t-1}^{t} \tilde{f}(\tau) d\tau, \]

\[ m_3(t) = 1 + \beta^{\frac{1}{1-a}} + \sup_{[t-1,t]} f^{\frac{2}{2-a}} + \int_{t-1}^{t} \tilde{f}(\tau) d\tau, \]

\[ A_1 = A + A^{\frac{2}{2-a}} + \limsup_{t \to \infty} \int_{t-1}^{t} \tilde{f}(\tau) d\tau. \]

Below,

\[ H(\xi) = \int_{0}^{\xi^2} K(\sqrt{s}) ds \quad \text{for} \ \xi \geq 0, \]

\[ J_H[u](t) = \int_{U} H(\|\nabla u(x, t)\|) dx \sim \int_{U} K(\|\nabla p\|) |\nabla p|^2 dx \sim \int_{U} |\nabla p|^{2-a} dx. \]
Estimates for derivatives

Theorem (H.-Ibragimov 2012)

Assume (DC).

(i) For all \( t \geq 1 \), one has

\[ J_H[p](t), \| \bar{p}_t(t) \|_{L^2}^2 \leq C m_1(t) \quad \text{for all } t \geq 1. \]

(iii) If \( A < \infty \) then there is \( T > 1 \) such that

\[ J_H[p](t), \| \bar{p}_t(t) \|_{L^2}^2 \leq C m_2(t) \quad \text{for all } t > T, \]

\[ \lim_{t \to \infty} J_H[p](t), \lim_{t \to \infty} \| \bar{p}_t(t) \|_{L^2}^2 \leq C A_1. \]

(iv) If \( \beta < \infty \) then there is \( T > 1 \) such that

\[ J_H[p](t), \| \bar{p}_t(t) \|_{L^2}^2 \leq C m_3(t) \quad \text{for all } t > T. \]
\( L^\infty \)-estimate - I. De Giorgi technique.

**Proposition (H.-Kieu-Phan 2013)**

Assume (SDC).

(i) Then

\[
\sup_{[0, T]} \| \bar{p} \|_{L^\infty} \leq C \left\{ \| \bar{p}_0 \|_{L^\infty} + (1 + T)^{\frac{2}{2-a}} \left( \sup_{[0, T]} \| \psi \|_{L^\infty} + 1 \right)^{\frac{1}{1-a}} \right\}.
\]

(ii) For any \( T_0 \geq 0, T > 0, \delta \in (0, 1] \) and \( \theta \in (0, 1) \),

\[
\sup_{[T_0 + \theta T, T_0 + T]} \| \bar{p} \|_{L^\infty} \leq C \left\{ \sqrt{\mathcal{E}} + (T + 1)^{\frac{n}{4-(n+2)a}} \left( 1 + \frac{1}{\delta^a \theta T} \right)^{\frac{n+2}{4-(n+2)a}} \cdot \left( \| \bar{p} \|_{L^2(U \times (T_0, T_0 + T))} + \| \bar{p} \|_{L^2(U \times (T_0, T_0 + T))}^{\frac{4}{4-(n+2)a}} \right) \right\},
\]

\[
\mathcal{E} = \mathcal{E}_{T_0, T} \overset{\text{def}}{=} \delta^{2-a} + T \delta^{-a} \sup_{[T_0, T_0 + T]} \| \psi \|_{L^\infty}^{2} + \delta^{-\frac{a(2-a)}{1-a}} \sup_{[T_0, T_0 + T]} \| \psi \|_{L^\infty}^{\frac{2-a}{1-a}}.
\]
\[ \| \bar{p}(t) \|_{L^\infty} \leq C \left( 1 + \| \bar{p}_0 \|_{L^\infty} + \| \bar{p}_0 \|_{L^2}^{\mu_4(2-a)} + M_f(t)^{\mu_4} \right) \]

\[ \| \bar{p}(t) \|_{L^\infty} \leq C \left( 1 + t^{-\frac{1}{\mu_2(2-a)}} \right) \left\{ 1 + \| \bar{p}_0 \|_{L^2}^{\mu_4(2-a)} + M_f(t)^{\mu_4} \right\} . \]

(ii) If \( A < \infty \) then \( \limsup_{t \to \infty} \| \bar{p}(t) \|_{L^\infty} \leq C \left( 1 + A^{\mu_4} \right) . \)

(iii) If \( \beta < \infty \) then there is \( T > 0 \) such that

\[ \| \bar{p}(t) \|_{L^\infty} \leq C \left\{ 1 + \beta^{\frac{\mu_4(2-a)}{2(1-a)}} + \sup_{[t-2,t]} \| \psi \|_{L^\infty}^{\frac{\mu_4(2-a)}{1-a}} \right\} \text{ for all } t > T . \]

(iv) If \( \lim_{t \to \infty} \| \psi(t) \|_{L^\infty} = 0 \) then \( \lim_{t \to \infty} \| \bar{p}(t) \|_{L^\infty} = 0 . \)
Gradient estimate

**Theorem (H.-Kieu-Phan 2013)**

Assume (SDC). For \( s \geq 2 \), \( U' \subseteq U \) and \( T > 1 \),

\[
\sup_{[0,T]} \int_{U'} |\nabla p(x, t)|^s dx \leq C L_4(s) \left( 1 + M_f(T) \right)^{\mu_4(s-2)} \left\{ 1 + \int_0^T f(t) dt \right\},
\]

where \( L_4(s) = \left( 1 + \|\bar{p}_0\|_{L^\infty} + \|\bar{p}_0\|_{L^2}^{\mu_4(2-a)} \right)^{s-2} \left\{ 1 + \int_U |\nabla p_0(x)|^s dx \right\}. \)

**Lemma (Ladyzhenskaya-Uraltseva-type embedding theorem)**

For each \( s \geq 1 \), smooth cut-off function \( \zeta(x) \in C_\infty(\U) \),

\[
\int_U K(|\nabla w|)|\nabla w|^{2s+2}\zeta^2 dx \leq C \sup_{\text{supp} \zeta} |w|^2 \left\{ \int_U K(|\nabla w|)|\nabla w|^{2s-2}|\nabla^2 w|^2 \zeta^2 dx \\
+ \int_U K(|\nabla w|)|\nabla w|^{2s}|\nabla \zeta|^2 dx \right\}.
\]
Dependence on the boundary data

Let $p_1(x, t)$ and $p_2(x, t)$ be two solutions having fluxes $\psi_1$ and $\psi_2$. Let $\Psi = \psi_1 - \psi_2$, $P = p_1 - p_2$, and $\bar{P} = P - |U|^{-1} \int_U P \, dx$.

**Notation.** We define for $i = 1, 2$,

$$f_i(t) = \|\psi_i(t)\|_{L^\infty}^{2-a} \|\psi_i(t)\|_{L^\infty}^{1-a}, \quad \tilde{f}_i(t) = \|\psi_{it}(t)\|_{L^\infty}^{2-a} \|\psi_{it}(t)\|_{L^\infty}^{1-a}.$$  

For $i = 1, 2$, we assume $f_i(t), \tilde{f}_i(t) \in C([0, \infty))$ and when needed $f_i(t) \in C^1((0, \infty))$; let

$$A_i = \limsup_{t \to \infty} f_i(t) \quad \text{and} \quad \beta_i = \limsup_{t \to \infty} [f_i'(t)]^-.$$  

Set $\bar{A} = A_1 + A_2$, $\bar{\beta} = \beta_1 + \beta_2$.

Let $F(t) = f_1(t) + f_2(t)$, $M_F(t) = M_{f_1}(t) + M_{f_2}(t)$, $\tilde{F}(t) = \tilde{f}_1(t) + \tilde{f}_2(t)$.

$$W(t) = 1 + M_F(t) + \int_{t-1}^t \tilde{F}(\tau) \, d\tau \quad \text{in general case, or}$$  

$$W(t) = 1 + \bar{\beta} + F(t) + F(t-1) + \int_{t-1}^t \tilde{F}(\tau) \, d\tau \quad \text{in case } \bar{\beta} < \infty.$$
Theorem (H.-Ibragimov 2012)

Assume (DC). (i) If $\bar{A} < \infty$ and $\int_1^\infty \left(1 + \int_T^{T-1} \tilde{F}(s)ds\right)^{-b} d\tau = \infty$ then

$$\limsup_{t \to \infty} \|\bar{P}(t)\|_{L^2}^2 \leq C \limsup_{t \to \infty} \left\{ \|\psi(t)\|_{L^\infty}^2 \left(1 + \bar{A} + \int_{t-1}^t \tilde{F}(\tau)d\tau\right)^{2b} \right\}.$$ 

(ii) If $\bar{A} = \infty$ and $\int_1^\infty W^{-b}(t)dt = \infty$ then

$$\limsup_{t \to \infty} \|\bar{P}(t)\|_{L^2}^2 \leq C \limsup_{t \to \infty} \{\|\psi(t)\|_{L^\infty}^2 W^{2b}(t)\}.$$ 

Theorem (H.-Kieu-Phan 2013)

Assume (SDC). For $T > 0$, we have

$$\sup_{[0,T]} \|\bar{P}\|_{L^\infty(U')} \leq 2\|\bar{P}(0)\|_{L^\infty} + CL_{11}M_{2,T} \left(\|\bar{P}(0)\|_{L^2} + \sup_{[0,T]} \|\psi(t)\|_{L^\infty} + \left[\|\bar{P}(0)\|_{L^2} + \sup_{[0,T]} \|\psi(t)\|_{L^\infty}\right]^{\gamma_1 \gamma_1 + 1}\right)$$
Theorem (H.-Ibragimov 2012)

Assume (DC).

(i) If $\bar{A} < \infty$ and $\int_{t-1}^{t} \tilde{F}(\tau) d\tau$ is uniformly bounded on $[1, \infty)$, then

$$\limsup_{t \to \infty} \|\nabla P(t)\|_{L^{2-a}}^{2} \leq CM_{4}^{2b+1/2} \limsup_{t \to \infty} \|\Psi(t)\|_{L^{\infty}}$$

$$+ CM_{4}^{2b} \limsup_{t \to \infty} \|\Psi(t)\|_{L^{\infty}}^{2},$$

where $M_{4} = 1 + \bar{A} + \limsup_{t \to \infty} \int_{t-1}^{t} \tilde{F}(\tau) d\tau$.

(ii) If $\bar{A} = \infty$ and $\lim_{t \to \infty} W'(t) W^{b-1}(t) = 0$ and $\int_{1}^{\infty} W^{-b}(\tau) d\tau = \infty$, then

$$\limsup_{t \to \infty} \|\nabla P(t)\|_{L^{2-a}}^{2} \leq C \limsup_{t \to \infty} \left( W^{2b+1/2}(t) \|\Psi(t)\|_{L^{\infty}} \right)$$

$$+ C \limsup_{t \to \infty} \left( W^{2b}(t) \|\Psi(t)\|_{L^{\infty}}^{2} \right).$$
Dependence on the Forchheimer polynomials

Recall

\[ g(|u|)u = -\nabla p, \]
\[ g(s) = a_0 s^{\alpha_0} + a_1 s^{\alpha_1} + a_2 s^{\alpha_2} + \ldots + a_N s^{\alpha_N}. \]

- Let \( N > 0 \) and \( \vec{\alpha} = (0, \alpha_1, \ldots, \alpha_N) \) be fixed. Denote

\[ R(N) = \{ \vec{a} = (a_0, a_1, \ldots, a_N) : a_0, a_N > 0, a_1, \ldots, a_{N-1} \geq 0 \}. \]

- Let \( D \) be a compact set in \( R(N) \).
- Let \( g_1(s) = g(s, \vec{a}^{(1)}) \) and \( g_2(s) = g(s, \vec{a}^{(2)}) \) be two functions of class \( \text{FP}(N, \vec{\alpha}) \), where \( \vec{a}^{(1)} \) and \( \vec{a}^{(2)} \) belong to \( D \).
- Let \( p_k = p_k(x, t; \vec{a}^{(k)}) \) be two solutions with the same flux \( \psi(x, t) \).
- Let \( P = p_1 - p_2, \bar{P} = \bar{p}_1 - \bar{p}_2 \).
**Theorem (H.-Ibragimov 2012)**

Assume (DC). If $f(t)$ and $\tilde{f}(t)$ are bounded, then

\[
\limsup_{t \to \infty} \| \tilde{P}(t) \|_{L^2}^2 \leq C_* M_8^{b+1} |\bar{a}(1) - \bar{a}(2)|,
\]

\[
\limsup_{t \to \infty} \| \nabla \tilde{P}(t) \|_{L^2-a}^2 \leq C_* M_8^{3b/2+1} |\bar{a}(1) - \bar{a}(2)|^{1/2} + C_* M_8^{b+1} |\bar{a}(1) - \bar{a}(2)|,
\]

where $M_8 = 1 + A + \limsup_{t \to \infty} \int_{t-1}^{t} \tilde{f}(\tau) d\tau$.

**Theorem (H.-Kieu-Phan 2013)**

Assume (SDC). Let $A = \| \tilde{P}(0) \|_{L^2} + |\bar{a}(1) - \bar{a}(2)|^{1/2}$.

\[
\sup_{[0,T]} \| \tilde{P} \|_{L^\infty(U')} \leq 2 \| \tilde{P}(0) \|_{L^\infty} + CM_6, T (A + A \frac{\gamma_1}{\gamma_1+1}), \quad T > 0,
\]

\[
\sup_{[2,\infty)} \| \tilde{P} \|_{L^\infty(U')} \leq C \gamma_8 (A + A \frac{\gamma_1}{\gamma_1+1}),
\]

\[
\limsup_{t \to \infty} \| \tilde{P}(t) \|_{L^\infty(U')} \leq C \gamma_9 \left( |\bar{a}(1) - \bar{a}(2)| + |\bar{a}(1) - \bar{a}(2)| \frac{\gamma_1}{\gamma_1+1} \right)^{1/2}.
\]
Part IIB. Two-phase incompressible fluids

For each $i$th-phase ($i = 1, 2$), saturation $S_i \in [0, 1]$, density $\rho_i \geq 0$, velocity $u_i \in \mathbb{R}^n$, and pressure $p_i \in \mathbb{R}$. The saturations satisfy

$$S_1 + S_2 = 1.$$  

Each phase’s velocity obeys the generalized Forchheimer equation. Conservation of mass holds for each of the phases:

$$\partial_t (\phi \rho_i S_i) + \text{div}(\rho_i u_i) = 0, \quad i = 1, 2.$$  

Due to incompressibility of the phases, i.e. $\rho_i = \text{const.} > 0$, it is reduced to

$$\phi \partial_t S_i + \text{div} \ u_i = 0, \quad i = 1, 2.$$  

Let $p_c$ be the capillary pressure between two phases, more specifically,

$$p_1 - p_2 = p_c.$$
Denote $S = S_1$ and $p_c = p_c(S)$. Then

$$g_i(|u_i|)u_i = -f_i(S)\nabla p_i, \quad i = 1, 2,$$

$$\nabla p_1 - \nabla p_2 = p'_c(S)\nabla S.$$ 

Hence

$$F_2(S)g_2(|u_2|)u_2 - F_1(S)g_1(|u_1|)u_1 = \nabla S,$$

where

$$F_i(S) = \frac{1}{p'_c(S)f_i(S)}, \quad i = 1, 2.$$ 

In summary,

$$0 \leq S = S(x, t) \leq 1,$$

$$S_t = -\text{div } u_1,$$

$$S_t = \text{div } u_2,$$

$$\nabla S = F_2(S)G_2(u_2) - F_1(S)G_1(u_1).$$
One-dimensional problem

Assumption A.

\[ f_1, f_2 \in C([0, 1]) \cap C^1((0, 1)), \]
\[ f_1(0) = 0, \quad f_2(1) = 0, \]
\[ f'_1(S) > 0, \quad f'_2(S) < 0 \text{ on } (0, 1). \]

Assumption B.

\[ p'_c \in C^1((0, 1)), \quad p'_c(S) > 0 \text{ on } (0, 1). \]

Theorem (H.-Kieu-Ibragimov 2013)

- There are 16 types of non-constant steady states (based on their monotonicity and asymptotic behavior as \( x \to \pm \infty \)).
- The steady states which are never 0 nor 1 are linearly stable.
Steady states with geometric constraints:

\[ u_1^*(x) = c_1 |x|^{-n} x, \quad u_2^*(x) = c_2 |x|^{-n} x, \quad S_*(x) = S(|x|), \]

where \( c_1, c_2 \) are constants and \( S(r) \) is a solution of the following ODE:

\[ S' = F(r, S(r)) \quad \text{for } r > r_0, \quad S(r_0) = s_0, \quad 0 < S(r) < 1. \]

where \( s_0 \) is always a number in \((0, 1)\) and

\[ F(r, S(r)) = G_2(c_2 r^{1-n}) F_2(S) - G_1(c_1 r^{1-n}) F_1(S). \]
Theorem

There exists a maximal interval of existence \([r_0, R_{\text{max}})\), where \(R_{\text{max}} \in (r_0, \infty]\), and a unique solution \(S \in C^1([r_0, R_{\text{max}}); (0, 1))\). Moreover, if \(R_{\text{max}}\) is finite then either

\[
\lim_{r \to R_{\text{max}}^-} S(r) = 0 \quad \text{or} \quad \lim_{r \to R_{\text{max}}^-} S(r) = 1.
\]

Theorem

If solution \(S(r)\) exists in \([r_0, \infty)\), then it eventually becomes monotone and, consequently, \(s_{\infty} = \lim_{r \to \infty} S(r)\) exists.

In case \(n = 2\) and \(c_1^2 + c_2^2 > 0\), let \(s^* = (f_1/f_2)^{-1}\left(\frac{c_1a_1^0}{c_2a_2^0}\right)\).

(i) If \(c_1 \leq 0\) and \(c_2 \geq 0\) then \(s_{\infty} = 1\).
(ii) If \(c_1 \geq 0\) and \(c_2 \leq 0\) then \(s_{\infty} = 0\).
(iii) If \(c_1, c_2 < 0\) then \(s_{\infty} = s^*\).
(iv) If \(c_1, c_2 > 0\) then \(s_{\infty} \in \{0, 1, s^*\} \).
The formal linearized system at the steady state \((u_1^*(x), u_2^*(x), S_*(x))\) is

\[
\sigma_t = -\text{div} \ v_1, \quad \sigma_t = \text{div} \ v_2,
\]

\[
\nabla \sigma = F_2(S_*)G'_2(u_2^*)v_2 + F'_2(S_*)\sigma G_2(u_2^*)
\]

\[
- \left( F_1(S_*)G'_1(u_1^*)v_1 + F'_1(S_*)\sigma G_1(u_1^*) \right).
\]

Let \(v = v_1 + v_2\). Then \(\text{div} \ v = 0\). Assume \(v = \mathbf{V}(x, t)\) is given. Let

\[
B = B(x) = F_2(S_*)G'_2(u_2^*) + F_1(S_*)G'_1(u_1^*),
\]

\[
A = A(x) = B(x)^{-1}
\]

\[
b = b(x) = F'_2(S_*)G_2(u_2^*) - F'_1(S_*)G_1(u_1^*)
\]

\[
c = c(x, t) = F_1(S_*)G'_1(u_1^*)\mathbf{V}(x, t).
\]

Decoupling the linearized system:

\[
\sigma_t = \nabla \cdot \left[ A(\nabla \sigma - \sigma b) \right] + \nabla \cdot (Ac),
\]

\[
v_2 = A(\nabla \sigma - \sigma b) + Ac, \quad v_1 = \mathbf{V} - v_2.
\]
In Bounded domains

**Condition (E1).** $F_1, F_2 \in C^7((0, 1))$ and $V \in C^6_x(D)$; $V_t \in C^3_x(D)$.

**Theorem**

Assume (E1) and $\Delta_4 \overset{\text{def}}{=} \sup_{D}(|V(x, t)| + |\nabla V(x, t)|) + \sup_{\Gamma \times [0, \infty)} |g(x, t)|$ is finite. Then the solution $\sigma(x, t)$ of the linearized equation satisfies

$$
\sup_{x \in U} |\sigma(x, t)| \leq C \left[ e^{-\eta_1 t} \sup_{U} |\sigma_0(x)| + \Delta_4 \right] \quad \text{for all } t > 0.
$$

Moreover,

$$
\limsup_{t \to \infty} \left[ \sup_{x \in U} |\sigma(x, t)| \right] \leq C \Delta_5,
$$

where

$$
\Delta_5 = \limsup_{t \to \infty} \left[ \sup_{x \in U} (|V(x, t)| + |\nabla V(x, t)|) + \sup_{x \in \Gamma} |g(x, t)| \right].
$$
Theorem

Assume (E1), and
\[ \Delta_6 \overset{\text{def}}{=} \sup_D \left( |V(x, t)| + |\nabla V(x, t)| + |\nabla^2 V(x, t)| \right) \]
and
\[ \Delta_7 \overset{\text{def}}{=} \sup_{\Gamma \times [0, \infty)} |g(x, t)| \]
are finite. Then for any \( U' \subseteq U \), there is \( \tilde{M} > 0 \) such that for \( i = 1, 2, x \in U' \) and \( t > 0 \),

\[ \sup_{x \in U'} |v_i(x, t)| \leq \tilde{M} \left( 1 + \frac{1}{\sqrt{t}} \right) \left[ e^{-\eta_1 t} \sup_U |\sigma_0(x)| + \Delta_6 + \sqrt{\Delta_6} + \Delta_7 \right]. \]

Consequently, if

\[ \lim_{t \to \infty} \left\{ \sup_{x \in U} (|V(x, t)| + |\nabla V(x, t)| + |\nabla^2 V(x, t)|) + \sup_{x \in \Gamma} |g(x, t)| \right\} = 0, \]

then for any \( x \in U \),

\[ \lim_{t \to \infty} v_1(x, t) = \lim_{t \to \infty} v_2(x, t) = 0. \]
Theorem (In unbounded domains)

Let $n \geq 3$. Assume $(\text{E1})$, $\Delta_{11} \overset{\text{def}}{=} \sup_D |\nabla \cdot (A(x)c(x, t))| < \infty$ and

$$\Delta_{10} \overset{\text{def}}{=} \max\left\{\sup_U |\sigma_0(x)|, \sup_{\Gamma \times [0, \infty)} |g(x, t)|\right\} < \infty.$$ 

(i) There exists a solution $\sigma(x, t) \in C_{x,t}^2(D) \cap C(\bar{D})$ such that

$$|\sigma(x, t)| \leq C \left[\Delta_{10} + \Delta_{11}(t + 1)\right].$$

(ii) If $\lim_{|x| \to \infty} \sigma_0(x) = 0$ and $\lim_{|x| \to \infty} \sup_{0 \leq t \leq T} |\nabla \cdot (A(x)c(x, t))| = 0$ for each $T > 0$, then

$$\lim_{r \to \infty} \left(\sup_{\{x:|x|=r\} \times [0, T]} |\sigma(x, t)|\right) = 0 \quad \text{for any } T > 0;$$

there is a continuous, increasing function $r(t) > 0$ with $\lim_{t \to \infty} r(t) = \infty$ such that

$$\lim_{t \to \infty} \left(\sup_{\{|x| \geq r(t)\}} |\sigma(x, t)|\right) = 0.$$


Scientific activities (since joining TTU in Fall 2008)

- Conference talks: 8
- TTU talks: 11
- Co-organizer of special sessions/mini-symposia: 3 in America, 1 in Italy
- Co-organizer of RedRaider Mini-symposia 2009 and 2013
- Co-organizer of Applied Mathematics Seminars (2008–present)
Acknowledgement

1 Collaborators:

- **TTU**: Eugenio Aulisa, Lidia Bloshanskaya, Emine Celik, Akif Ibragimov, Thinh Kieu, and in project discussion with David Gilliam, Alex Yu. Solynin.

2 NSF-Grants:

THANK YOU FOR YOUR ATTENTION!