Forchheimer Equations in Porous Media - Part I

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Introduction

- Darcy's Law:

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- the "two term" law

$$\alpha u + \beta |u|_B \, u = -\Pi \nabla p,$$

- the "power" law

$$c^n |u|_B^{n-1} \, u + a u = -\Pi \nabla p,$$

- the "three term" law

$$\mathcal{A} u + \mathcal{B} \, |u| \, u + \mathcal{C} |u|_B^2 \, u = -\Pi \nabla p.$$

Here α, β, c, $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ are empirical positive constants.

Above $B = B(x)$ is positive definite, $\Pi = \Pi(x)$ is the (normalized) permeability tensor, positive definite, symmetric, and satisfies

$$k_1 \geq y^{\mathsf{T}} \Pi \; y / |y|^2 \geq \; k_0 > 0,$$

the norms are $|y| = (\sum_{i=1}^d y_i^2)^{1/2}$ and $|u|_B = \sqrt{(u^{\mathsf{T}} B u)}$.

General Forchheimer equations

Generalizing the above equations as follows

$$g(x, |u|_B)u = -\Pi \nabla p.$$

Let $B = I_3$, $|u|_B = |u|$, $g = g(|u|)$, and $\Pi = I_3$. Solve for u in terms of ∇p.

Let $G(s) = sg(s)$. Then $G(|u|) = |\nabla p| \Rightarrow |u| = G^{-1}(|\nabla p|)$. Hence

$$u = -\frac{\nabla p}{g(G^{-1}(|\nabla p|))} = -K(|\nabla p|)\nabla p,$$

where

$$K(\xi) = K_g(\xi) = \frac{1}{g(s)} = \frac{1}{g(G^{-1}(\xi))}, \quad sg(s) = \xi.$$

We derived non-linear Darcy equations from Forchheimer equations.

Equations of Fluids

Let ρ be the density. Continuity equation

$$\frac{d\rho}{dt} = -\nabla \cdot (\rho u),$$

For slightly compressible fluid it takes

$$\frac{d\rho}{dp} = \frac{1}{\kappa}\rho,$$

where $\kappa \gg 1$. Substituting this into the continuity equation yields

$$\frac{d\rho}{dp}\frac{dp}{dt} = -\rho\nabla \cdot u - \frac{d\rho}{dp}u \cdot \nabla p,$$

$$\frac{dp}{dt} = -\kappa\nabla \cdot u - u \cdot \nabla p.$$

Since $\kappa \gg 1$, we neglect the second term in continuity equation

$$\frac{dp}{dt} = -\kappa\nabla \cdot u\,.$$

Non-dimensional Equations and Boundary Conditions

Combining the equation of pressure and the Forchheimer equation, one gets after scaling:

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Consider the equation on a bounded domain U in $\mathbb{R}^3$. The boundary of U consists of two connected components: exterior boundary Γ_e and interior (accessible) boundary Γ_i.

- On Γ_e:

$$u \cdot N = 0 \Leftrightarrow \frac{\partial p}{\partial N} = 0.$$

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- On Γ_e:

$$u \cdot N = 0 \Leftrightarrow \frac{\partial p}{\partial N} = 0.$$

- Dirichlet condition on Γ_i: $p(x,t) = \phi(x,t)$ which is known for $x \in \Gamma_i$.
- Total flux condition on Γ_i:

$$\int_{\Gamma_i} u \cdot N d\sigma = Q(t) \Leftrightarrow \int_{\Gamma_i} K(|\nabla p|) \nabla p \cdot N d\sigma = -Q(t),$$

  where $Q(t)$ is known.

Monotonicity and Uniqueness of IBVP

Let F be a mapping from $\mathbb{R}^3$ to $\mathbb{R}^3$.

- F is (positively) monotone if

$$(F(y') - F(y)) \cdot (y' - y) \geq 0, \text{ for all } y', y \in \mathbb{R}^d.$$

- F is strictly monotone if there is $c > 0$ such that

$$(F(y') - F(y)) \cdot (y' - y) \geq c|y' - y|^2, \text{ for all } y', y \in \mathbb{R}^d.$$

- F is strictly monotone on bounded sets if for any $R > 0$, there is a positive number $c_R > 0$ such that

$$(F(y') - F(y)) \cdot (y' - y) \geq c_R|y' - y|^2, \text{ for all } |y'| \leq R, \ |y| \leq R.$$

Existence of function $K(\cdot)$

Aim $K_g(\xi) = 1/g(G^{-1}(\xi))$ exists.
G-Conditions:

$$g(0) > 0, \quad \text{and} \quad g'(s) \geq 0 \text{ for all } s \geq 0.$$

Lemma

Let $g(s)$ satisfy the G-Conditions. Then $K(\xi)$ exists. Moreover, for any $\xi \geq 0$, one has

$$K'(\xi) = -K(\xi)\frac{g'(s)}{\xi g'(s) + g^2(s)} \leq 0,$$

$$(K(\xi)\xi^n)' = K(\xi)\xi^{n-1}\left(n - \frac{\xi g'(s)}{\xi g'(s) + g^2(s)}\right) \geq 0,$$

for any $n \geq 1$, where $s = G^{-1}(\xi)$.

Aim: $F(y) = K(|y|)y$ is monotone.
Lambda-Condition:

$$g(s) \geq \lambda s g'(s), \quad \text{some} \quad \lambda > 0.$$

Proposition

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Proposition

Let $g(s)$ satisfy the G-Conditions. Then $F(y) = K(|y|)y$ is monotone. In addition, if g satisfies the Lambda-Condition, then $F(y)$ is strictly monotone on bounded sets. More precisely,

$$(F(y) - F(y')) \cdot (y - y') \geq \frac{\lambda}{\lambda + 1} K(\max\{|y|, |y'|\})|y' - y|^2.$$

Class (APPC)

We introduce a class of "algebraic polynomials with positive coefficients"

Definition

A function $g(s)$ is said to be of class (APPC) if

$$g(s) = a_0 s^{\alpha_0} + a_1 s^{\alpha_1} + a_2 s^{\alpha_2} + \ldots + a_k s^{\alpha_k} = \sum_{j=0}^{k} a_j s^{\alpha_j},$$

where $k \geq 0$, $0 = \alpha_0 < \alpha_1 < \alpha_2 < \ldots < \alpha_k$, and $a_0, a_1, \ldots, a_k$ are positive coefficients.

Proposition

Let $g(s)$ be a function of class (APPC). Then g satisfies G-Conditions and Lambda-Condition. Consequently, $F(y) = K_g(|y|)y$ is strictly monotone on bounded sets.

Uniqueness of solutions to IBVP-I

Proposition

Let $g(s)$ satify the G-Conditions. Let p_1 and p_2 are two solutions of IBVP with the same Dirichlet condition on Γ_i. Then

$$\int_U |p_1(x,t) - p_2(x,t)|^2 dx \le \int_U |p_1(x,0) - p_2(x,0)|^2 dx.$$

Consequently, if $p_1(x,0) = p_2(x,0) = p_0(x) \in L^2(U)$, then $p_1(x,t) = p_2(x,t)$ for all t.

Asymptotic Stability for IBVP-I

Proposition

Assume additionally that $g(s)$ satisfies the Lambda-Condition, and

$$\nabla p_1, \nabla p_2 \in L^\infty(0,\infty; L^\infty(U)),$$

then

$$\int_U |p_1(x,t) - p_2(x,t)|^2 dx \le e^{-c_1 K(M)t} \int_U |p_1(x,0) - p_2(x,0)|^2 dx,$$

for all $t \ge 0$, where

$$M = \max\{\|\nabla p_1\|_{L^\infty(0,T,L^\infty(U))}, \|\nabla p_2\|_{L^\infty(0,T;L^\infty(U))}\}.$$

Consequently,

$$\lim_{t\to\infty} \int_U |p_1(x,t) - p_2(x,t)|^2 dx = 0.$$

IBVP-II

Proposition

Let $g(s)$ satify the G-Conditions. Let p_1 and p_2 are two solutions of IBVP with the same total flux $Q(t)$ on Γ_i.
Assume that $(p_1 - p_2)|_{\Gamma_i}$ is independent of the spatial variable x.
Then

$$\int_U |p_1(x,t) - p_2(x,t)|^2 dx \le \int_U |p_1(x,0) - p_2(x,0)|^2 dx.$$

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Similar results to the Dirichlet condition hold when g satisfies the Lambda-Condition.

Pseudo Steady State Solutions

Definition

The solution $\overline{p}(x,t)$ is called a pseudo steady state (PSS) if

$$\frac{\partial \overline{p}(x,t)}{\partial t} = const, \quad \text{for all} \quad t.$$

This leads to the equation

$$\frac{\partial \overline{p}(x,t)}{\partial t} = -A = \nabla \cdot (K(|\nabla \overline{p}|)\nabla \overline{p}),$$

where A is a constant.
Integrating the equation over U gives

$$A|U| = -\int_{\Gamma_i} (K(|\nabla \overline{p}|)\nabla \overline{p}) \cdot N d\sigma = \int_{\Gamma_i} u \cdot N d\sigma = Q(t).$$

Therefore, the total flux of a PSS solution is

$$Q(t) = A|U| = Q, \quad \text{for all} \quad t.$$

PSS Solutions

Write

$$\overline{p}(x,t) = -At + h(x),$$

one has $\nabla p = \nabla h$, hence h and p satisfy the same PDE and boundary condition on Γ_e. On Γ_i, in general, we consider

$$h(x) = \varphi(x) \quad \text{on} \quad \Gamma_i.$$

In the case $\varphi(x) = const.$ on Γ_i, by shifting has

$$\overline{p}(x,t) = -At + B + W(x),$$

where A and B are two numbers, and $W(x) = W_A(x)$ satisfies

$$-A = \nabla \cdot (\, K(|\nabla W|)\nabla W\,),$$

$$\frac{\partial W}{\partial N} = 0 \quad \text{on} \quad \Gamma_e,$$

$$W = 0 \quad \text{on} \quad \Gamma_i,$$

Estimation of PSS Solutions

Proposition

Let $g(s)$ belong to class (APPC). Then for any number A, the basic profile $W = W_A$ satisfies

$$\|\nabla W\|_{L^{2-a}} \leq M = C(|A| + 1)^{1/(1-a)},$$

where $a = \frac{\alpha_k}{\alpha_k+1}$.

Continuous dependence of PSS Solutions on the total flux

Proposition

Let $g(s)$ be of class (APPC). Then there exists constant C such that for any $A_1,\ A_2$, the corresponding profiles $W_1,\ W_2$ satisfy

$$\left(\int_U |\nabla(W_1 - W_2)|^{2-a}\, dx\right)^{\frac{2}{2-a}} \leq C\, M\, |A_1 - A_2| \int_U |W_1 - W_2| dx,$$

where $M = (\max(|A_1|, |A_2|) + 1)^{a/(1-a)}$ and $C > 0$ is independent of A_1 and A_2.

Continuous dependence of PSS Solutions on the total flux

Corollary

Under the same assuptions as the previous proposition, there exists a constant C such that

$$\|\nabla(W_1 - W_2)\|_{L^{2-a}} \le C \cdot M \cdot |A_1 - A_2|,$$

where $M = [\max(|A_1|, |A_2|) + 1]^{a/(1-a)}$.