# Navier–Stokes equations with Navier boundary conditions in nearly flat domains

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# Outline

# Introduction

Navier-Stokes equations (NSE) in $\mathbb{R}^3$ with a potential body force

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u = -\nabla p + f, \\ \operatorname{div}\ u = 0, \\ u(x,0) = u_0(x), \end{cases}$$

$\nu > 0$ is the kinematic viscosity,
$u = (u_1, u_2, u_3)$ is the unknown velocity field,
$p \in \mathbb{R}$ is the unknown pressure,
$f(t)$ is the body force,
$u_0$ is the initial velocity.

# Navier boundary conditions

On the boundary $\partial\Omega$:

$$u \cdot N = 0,$$

$$\nu[D(u)N]_{\mathrm{tan}} = 0,$$

where $N$ is the outward normal vector, $[\,\cdot\,]_{\mathrm{tan}}$ denotes the tangential part,

$$D(u) = \frac{1}{2}\Big(\nabla u + (\nabla u)^*\Big).$$

We assume: $\nu = 1$.
Note: if the bounadry is flat, say, part of $x_3 = const$, then the condiontions become the free boundary condition

$$u_3 = 0, \quad \partial_3 u_1 = \partial_3 u_2 = 0.$$

Works by Temam-Ziane, primitive equations.

# Thin domains

$$\Omega = \Omega^\varepsilon = \{(x_1, x_2, x_3) \,:\, (x_1, x_2) \in \mathbb{T}^2,\, h_0^\varepsilon(x_1, x_2) < x_3 < h_1^\varepsilon(x_1, x_2)\},$$

where $\varepsilon \in (0, 1]$,

$$h_0^\varepsilon = \varepsilon g_0, \quad h_1^\varepsilon = \varepsilon g_1,$$

and $g_0, g_1$ are given $C^4$ functions defined on $\mathbb{T}^2$.
The boundary is $\Gamma = \Gamma_0 \cup \Gamma_1$, where $\Gamma_0$ is the bottom and $\Gamma_1$ is the top.
We define

$$M_0\phi(x') = \frac{1}{h_1 - h_0} \int_{h_0}^{h_1} \phi(x', x_3) dx_3,$$

$$\widehat{M} u = (M_0 u_1, M_0 u_2, 0).$$

# Main results

## Theorem (Global Existence Theorem)

*There are $\kappa > 0$ and $\varepsilon_0 > 0$ such that if $0 < \varepsilon \le \varepsilon_0$, $u_0 \in V$ and $f$ :*

$$\begin{aligned} \|u_0\|_{H^1}^2 &\le \kappa^2 \varepsilon^{-1}, & \|\widehat{M} u_0\|_{L^2}^2 &\le \kappa^2, \\ \|Pf\|_{L^\infty L^2}^2 &\le \kappa^2 \varepsilon^{-1}, & \|\widehat{M}(Pf)\|_{L^\infty L^2}^2 &\le \kappa^2 \end{aligned} \tag{1}$$

*then there exists a unique, globally defined strong solution $u = u(t)$ of the Navier-Stokes equations, with $u(0) = u_0$:*

$$u \in C^0([0,\infty); H^1(\Omega^\varepsilon)) \cap L^\infty((0,\infty); H^1(\Omega^\varepsilon)) \cap L^2_{\mathrm{loc}}([0,\infty); H^2(\Omega^\varepsilon)).$$

*Also, one has*

$$\|u(t)\|_{H^1}^2 \le \varepsilon^{-1} \left( M_1^2 e^{-2\alpha t} + L_1^2 \right), \qquad \text{for all } t \ge 0,$$

*where $M_1, L_1, \alpha > 0$.*

### Theorem (Global Attractor)

*Suppose $f$ is independent of $t$, and $f$ satisfies the above condition. Then the global attractor $\mathcal{A}$ for the above strong solutions also attracts all Leray-Hopf weak solutions of the Navier–Stokes equations .*

# A Green's formula

$$\int_\Omega \Delta u \cdot v \, dx = \int_\Omega [-2(Du : Dv) + (\nabla \cdot u)(\nabla \cdot v)] \, dx$$
$$+ \int_{\partial\Omega} \{2((Du)N) \cdot v - (\nabla \cdot u)(v \cdot N)\} \, d\sigma.$$

If $u$ is divergence-free and satisfies the Navier boundary condition, $v$ is tangential to the boundary then

$$-\int_\Omega \Delta u \cdot v \, dx = 2 \int_\Omega (Du : Dv) \, dx.$$

# Uniform Korn inequality

Let $H_0 = \{(a_1, a_2, 0) : a_1\partial_1 g + a_2\partial_2 g = 0\}$, where $g = g_1 - g_0$.

**Lemma**

*Let $u \in H^1(\Omega^\varepsilon) \cap H_0^\perp$, $u$ is tangential to the boundary of $\Omega^\varepsilon$. Then*

$$C_1\|u\|_{H^1} \leq \|Du\|_{L^2} \leq C_2\|u\|_{H^1},$$

*for $\varepsilon \in (0, 1]$, and $C_1, C_2$ are positive constant independent of $\varepsilon$.*

# Boundary conditions

## Lemma

*Let $\tau$ be a tangential vestor field on the boundary. If $u$ satisfies the Navier boundary conditions then*

$$\frac{\partial u}{\partial \tau} \cdot N + u \cdot \frac{\partial N}{\partial \tau} = 0,$$

$$\frac{\partial u}{\partial N} \cdot \tau = u \cdot \frac{\partial N}{\partial \tau}.$$

Combining with the trace theorem on the thin domain:

$$\|u_3\|_{L^2} \leq C\varepsilon \|u\|_{H^1},$$

$$\|\partial_3 u_1\|_{L^2} \leq C\varepsilon \|u\|_{H^2}, \quad \|\partial_3 u_2\|_{L^2} \leq C\varepsilon \|u\|_{H^2}.$$

# The Stokes operator

Leray-Helmholtz decomposition

$$L^2(\Omega^\varepsilon)^3 = H \oplus H_0 \oplus H_1^\perp = H_1 \oplus H_1^\perp,$$

where $H_1^\perp = \{\nabla\phi : \phi \in H^1(\Omega^\varepsilon)\}$.

$$\mathcal{D}(A) = \{u \in H^2(\Omega^\varepsilon) \cap H : \text{ u satisfies the Navier boundary conditions}\}.$$

Let $P$ denotes the (Leray) projection on $H$. Then the Stokes operator is:

$$Au = -P\Delta u, \quad u \in \mathcal{D}(A).$$

**Lemma**

*If $\varepsilon$ is small and $u \in \mathcal{D}(A)$ then*

$$\|Au + \Delta u\|_{L^2} \le C_1(\varepsilon\|\nabla u\|_{L^2} + \|u\|_{L^2}),$$

$$C_2\|Au\|_{L^2} \le \|u\|_{H^2} \le C_3\|Au\|_{L^2},$$

*where $C_1, C_2, C_3$ are independent of $\varepsilon$.*

Also, if $u \in V = \mathcal{D}(A^{1/2})$ then

$$C_1 \|A^{1/2} u\|_{L^2} \leq \|u\|_{H^1} \leq C_5 \|A^{1/2} u\|_{L^2}.$$

# Non-linear estimate

## Proposition

For any $\varepsilon \in (0,1]$, $u \in \mathcal{D}(A)$ and $\beta > 0$, one has

$$|\langle (u \cdot \nabla)u, Au \rangle| \leq \beta \|u\|_{H^2}^2 + C\varepsilon^{1/2}\|u\|_{H^1}\|u\|_{H^2}^2 \\ + C_\beta \varepsilon^{-1}\|u\|_{L^2}^2\|u\|_{H^1}^2,$$

where $C > 0$ is independent of $\beta$ and $\varepsilon$; and $C_\beta > 0$ is independent of $\varepsilon$.

# Non-linear estimate

## Proposition

For any $\varepsilon \in (0,1]$, $u \in \mathcal{D}(A)$ and $\beta > 0$, one has

$$|\langle (u\cdot\nabla)u, Au\rangle| \le \beta\,\|u\|_{H^2}^2 + C\varepsilon^{1/2}\|u\|_{H^1}\|u\|_{H^2}^2 + C_\beta\varepsilon^{-1}\|u\|_{L^2}^2\|u\|_{H^1}^2,$$

where $C > 0$ is independent of $\beta$ and $\varepsilon$; and $C_\beta > 0$ is independent of $\varepsilon$.

## Corollary

*There is $\varepsilon_* \in (0,1]$ such that for any $\varepsilon < \varepsilon_*$, $u \in \mathcal{D}(A)$ and any $\beta > 0$ we have*

$$|\langle (u\cdot\nabla)u, Au\rangle| \le \beta\,\|Au\|_{L^2}^2 + C\varepsilon^{1/2}\|A^{1/2}u\|\|Au\|_{L^2}^2 + C_\beta\varepsilon^{-1}\|u\|_{L^2}^2\|A^{1/2}u\|^2,$$

*where $C > 0$ is independent of $\beta$ and $\varepsilon$; and $C_\beta > 0$ is independent of $\varepsilon$.*

## Sketch of the proof

Averaging operator:

$$Mu = \Big(M_0 u_1, M_0 u_2, (M_0 u_1, M_0 u_2) \cdot \psi\Big),$$

where $\psi$ is determined so that $Mu \in H_1$ whenver $u \in H_1$.
Let $v = Mu$ and $w = u - v$.
Establish Ladyzhenskaya inequality for $v$.
Establish Agmon, Poincare, Ladyzhenskaya, Galiardo-Nirengberg, Sobolev inequalities for $w$.

$$\langle u \cdot \nabla u, Au\rangle = \langle w \cdot \nabla u, Au\rangle + \langle v \cdot \nabla u, Au + \Delta u\rangle - \langle v \cdot \nabla u, \Delta u\rangle.$$

Integration by parts: $-\langle v \cdot \nabla u, \Delta u\rangle = I_2 + I_3$, where

$$I_3 = \int_\Gamma (v \cdot \nabla) u \frac{\partial u}{\partial N} d\sigma.$$

# Proof (continued)

Let $b = (v \cdot \nabla)u = b_1\tau_1 + b_2\tau_2 + b_3 N$.

$$|b_1\tau_1 \cdot \frac{\partial u}{\partial N}| = |b_1 u \cdot \frac{\partial N}{\partial \tau}| \le C\varepsilon|v||\nabla u||u|.$$

$$|b_3 N \cdot \frac{\partial u}{\partial N}| \le C|b_3||\nabla u|.$$

Noting that $v = v_1\tau_1 + v_2\tau_2$,

$$|b_3| = |(\nabla u)v \cdot N| = |-(\nabla N)v \cdot u)| = |u \cdot (v_1\frac{\partial N}{\partial \tau_1} + v_2\frac{\partial N}{\partial \tau_2})| \le \varepsilon|u||v|.$$

Hence

$$I_3 \le \varepsilon \int_\Gamma |u||v||\nabla u| d\sigma.$$

# Global sotutions

Note $\|(I-\widehat{M})u\|_{L^2} \le C\varepsilon\|u\|_{H^1}$.

$L^2$-Estimate.

$$\begin{aligned} |\langle f, u\rangle| &= |\langle (I-\widehat{M})Pf, (I-\widehat{M})u\rangle + \langle \widehat{M}Pf, \widehat{M}u\rangle| \\ &= C\varepsilon\|Pf\|_{L^2}\|u\|_{H^1} + \|\widehat{M}Pf\|_{L^2}\|u\|_{L^2}. \end{aligned}$$

Then

$$\frac{d}{dt}\|u\|_{L^2}^2 + 2\alpha\|A^{1/2}u\|_{L^2}^2 \le C\|\widehat{M}Pf\|_{L^2}^2 + C\varepsilon^2\|Pf\|_{L^2}^2.$$

$$\|u_0\|_{L^2}^2 = \|\widehat{M}u_0\|_{L^2}^2 + \|(I-\widehat{M})u_0\|_{L^2}^2 \le \|\widehat{M}u_0\|_{L^2}^2 + C\varepsilon^2\|u_0\|_{H^1}^2.$$

$$\|u(t)\|_{L^2}^2, \int_t^{t+1} \|A^{1/2}u\|_{L^2}^2 ds \le C\kappa^2(e^{-2\alpha t} + 1).$$

$H^1$-estmimate

$$\frac{1}{2}\frac{d}{dt}\|A^{1/2}u\|_{L^2}^2+\|Au\|_{L^2}^2 \leq \beta\|u\|_{H^2}^2+C\varepsilon^{1/2}\|u\|_{H^1}\|u\|_{H^2}^2+C_\beta\varepsilon^{-1}\|u\|_{L^2}^2\|u\|_{H^1}^2+\|$$

$$\frac{d}{dt}\|A^{1/2}u\|_{L^2}^2+(1-C\varepsilon^{1/2}\|A^{1/2}u\|_{L^2})\|Au\|_{L^2}^2 \leq C\varepsilon^{-1}\|u\|_{L^2}^2\|A^{1/2}u\|_{L^2}^2+C\|Pf\|_L^2$$

As far as $1 - C\varepsilon^{1/2}\|A^{1/2}u\|_{L^2} < 1/2$, say in $[0, T)$ then by the Uniform Gronwall Inequality one has for $1 < t < T$:

$$\|A^{1/2}u(t)\|_{L^2}^2 \leq \int_{t-1}^{t} C\varepsilon^{-1}\|u\|_{L^2}^2\|A^{1/2}u\|_{L^2}^2+C\|Pf\|_{L^2}^2+\|A^{1/2}u\|_{L^2}^2 ds \leq C\kappa^2\varepsilon^{-}$$

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