The Normal Form of the Navier–Stokes equations in Suitable Normed Spaces

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Outline

1. Introduction
2. Main Results
3. Sketch of the Proof
4. Open Problems
Navier-Stokes equations (NSE) in $\mathbb{R}^3$ with a potential body force

\[
\begin{cases}
\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u = -\nabla p - \nabla \phi, \\
\text{div } u = 0, \\
u > 0 \text{ is the kinematic viscosity}, \\
\text{div } u = 0, \\
u > 0 \text{ is the kinematic viscosity},
\end{cases}
\]

$\nu > 0$ is the kinematic viscosity,

$u = (u_1, u_2, u_3)$ is the unknown velocity field,

$p \in \mathbb{R}$ is the unknown pressure,

$\phi$ is the potential of the body force,

$u^0$ is the initial velocity.
Let $L > 0$ and $\Omega = (0, L)^3$. The $L$-periodic solutions:

$$u(x + Le_j) = u(x) \text{ for all } x \in \mathbb{R}^3, j = 1, 2, 3,$$

where $\{e_1, e_2, e_3\}$ is the canonical basis in $\mathbb{R}^3$.

Zero average condition

$$\int_{\Omega} u(x) dx = 0,$$

Throughout $L = 2\pi$ and $\nu = 1$. 
The Stokes operator:

\[ Au = -\Delta u \text{ for all } u \in \mathcal{D}_A. \]

The bilinear mapping:

\[ B(u, v) = P_L(u \cdot \nabla v) \text{ for all } u, v \in \mathcal{D}_A. \]

\( P_L \) is the Leray projection from \( L^2(\Omega) \) onto \( H \).

Spectrum of \( A \):

\[ \sigma(A) = \{|k|^2, 0 \neq k \in \mathbb{Z}^3\}. \]

If \( N \in \sigma(A) \), denote by \( R_N H \) the eigenspace of \( A \) corresponding to \( N \).
Otherwise, \( R_N H = \{0\} \).
Denote by $\mathcal{R}$ the set of all initial data $u^0 \in V$ such that the solution is regular for all times $t > 0$. In particular $u(t) \in \mathcal{D}_A$ for all $t > 0$.

The functional form of the NSE:

$$\frac{du(t)}{dt} + Au(t) + B(u(t), u(t)) = 0, \quad t > 0,$$

$$u(0) = u^0 \in \mathcal{R},$$

where the equation holds in $\mathcal{D}_A$ for all $t > 0$ and $u(t)$ is continuous from $[0, \infty)$ into $V$. 
Asymptotic expansion of regular solutions

Asymptotic expansion of $u(t) = u(t, u^0)$ (Foias-Saut)

$$u(t) \sim q_1(t)e^{-t} + q_2(t)e^{-2t} + q_3(t)e^{-3t} + ...,$$

where $q_j(t) = W_j(t, u^0)$ is a polynomial in $t$ of degree at most $(j - 1)$ and with values are trigonometric polynomials. This means that for any $N \in \mathbb{N}$,

$$|u(t) - \sum_{j=1}^{N} q_j(t)e^{-jt}| = O(e^{-(N+\varepsilon)t}) \text{ for } t \to \infty,$$

with some $\varepsilon = \varepsilon_N > 0$. Moreover (Guillope), for $m \in \mathbb{N}$,

$$\|u(t) - \sum_{j=1}^{N} q_j(t)e^{-jt}\|_{H^m(\Omega)} = O(e^{-(N+\varepsilon)t})$$

as $t \to \infty$, for some $\varepsilon = \varepsilon_{N,m} > 0$.
Normalization map

Let

\[ W(u^0) = W_1(u^0) \oplus W_2(u^0) \oplus \cdots, \]

where \( W_j(u^0) = R_j q_j(0) \), for \( j = 1, 2, 3 \ldots \). Then \( W \) is an one-to-one analytic mapping from \( \mathcal{R} \) to the Frechet space

\[ S_A = R_1 H \oplus R_2 H \oplus \cdots. \]
Constructions of polynomials $q_j(t)$

If $u^0 \in \mathcal{R}$ and $W(u^0) = (\xi_1, \xi_2, ...)$, then $q_j$’s are the unique polynomial solutions to the following equations

$$q'_j + (A - j)q_j + \beta_j = 0,$$

with $R_j q_j(0) = \xi_j$, where $\beta_j$’s are defined by

$$\beta_1 = 0 \text{ and for } j > 1, \beta_j = \sum_{k+l=j} B(q_k, q_l).$$

Explicitly, these polynomials $q_j(t)$’s are recurrently given by

$$q_j(t) = \xi_j - \int_0^t R_j \beta_j(\tau) d\tau$$

$$+ \sum_{n \geq 0} (-1)^{n+1} [(A - j)(I - R_j)]^{-n-1} \left( \frac{d}{dt} \right)^n (I - R_j) \beta_j,$$

where $[(A - j)(I - R_j)]^{-n-1} u(x) = \sum_{|k|^2 \neq j} \frac{a_k}{(|k|^2 - j)^{n+1}} e^{ik \cdot x}$, for

$$u(x) = \sum_{|k|^2 \neq j} a_k e^{ik \cdot x} \in \mathcal{V}.$$
The $S_A$-valued function $\xi(t) = (\xi_n(t))_{n=1}^{\infty} = (W_n(u(t)))_{n=1}^{\infty} = W(u(t))$ satisfies the following system of differential equations

$$\frac{d \xi_1(t)}{dt} + A \xi_1(t) = 0,$$

$$\frac{d \xi_n(t)}{dt} + A \xi_n(t) + \sum_{k+j=n} R_n B(q_k(0, \xi(t)), q_j(0, \xi(t))) = 0, \quad n > 1.$$ 

The solution of the above system with initial data $\xi^0 = (\xi_n^0)_{n=1}^{\infty} \in S_A$ is precisely $(R_n q_n(t, \xi^0) e^{-nt})_{n=1}^{\infty}$. 

(Foias-Hoang-Olson-Ziane)
A construction of regular solutions

Split the initial data $u^0$ in $V$ as $u^0 = \sum_{n=1}^{\infty} u^0_n$. We find the solution $u(t)$ of the form $u(t) = \sum_{n=1}^{\infty} u_n(t)$, where for each $n$,

$$\frac{du_n(t)}{dt} + Au_n(t) + B_n(t) = 0, \quad t > 0,$$

with initial condition

$$u_n(0) = u^0_n,$$

where

$$B_1(t) \equiv 0, \quad B_n(t) = \sum_{j+k=n} B(u_j(t), u_k(t)), \quad n > 1.$$

We call the above system the extended Navier–Stokes equations.
Existence theorems

Theorem (2006)

Let $S^0 = \sum_{n=1}^{\infty} \|u_n^0\| < \varepsilon_0$ and $u(t) = \sum_{n=1}^{\infty} u_n(t)$.

If $S^0$ is small then $u(t)$, $t \geq 0$, is the unique solution of the Navier–Stokes equations where $u^0 = \sum_{n=1}^{\infty} u_n^0 \in V$ and

$$\sum_{n=1}^{\infty} \|u_n(t)\| \leq 2S^0 e^{-t}, \quad t > 0.$$  

If $S^0 = \sum_{n=1}^{\infty} \|u_n^0\| < \infty$, then $u(t) = \sum_{n=1}^{\infty} u_n(t)$ is the regular solution in $(0, T)$ for some $T > 0$. 

(Foias-Hoang-Olson-Ziane)
Connection to the asymptotic expansions

Theorem (2006)

Suppose \( \sum_{n=1}^{\infty} \|W_n(0, u^0)\| < \varepsilon_0 \), then \( u(t, u^0) = \sum_{n=1}^{\infty} W_n(t, u^0)e^{-nt} \) is the regular solution to the Navier–Stokes equations for all \( t > 0 \).

Theorem (2006)

Suppose \( \lim \sup_{n \to \infty} \|W_n(0, u^0)\|^{1/n} < \infty \). Then there is \( T > 0 \) such that

\[
\nu(t) = \sum_{n=1}^{\infty} W_n(t, u^0)e^{-nt}
\]

is absolutely convergent in \( V \), uniformly in \( t \in [T, \infty) \), \( \sum_{n=1}^{\infty} W_n(t, u^0)e^{-nt} \) is the asymptotic expansion of \( \nu(t) \), and

\[
u(t, u^0) = \nu(t) \text{ for all } t \in [T, \infty).
\]
Algebraic relations

Let $V^\infty = V \oplus V \oplus V \oplus \cdots$. Define

$$W(t, \cdot) : u \in \mathcal{R} \mapsto (W_n(t, u)e^{-nt})_{n=1}^\infty \in V^\infty,$$

$$Q(t, \cdot) : \bar{\xi} \in S_A \mapsto (q_n(t, \bar{\xi})e^{-nt})_{n=1}^\infty \in V^\infty.$$

We primarily have
Let \((\tilde{\kappa}_n)_{n=2}^\infty\) be a fixed sequence of real numbers in the interval \((0, 1]\) satisfying
\[
\lim_{n \to \infty} (\tilde{\kappa}_n)^{1/2^n} = 0.
\]
We define the sequence of positive weights \((\rho_n)_{n=1}^\infty\) by
\[
\rho_1 = 1, \quad \rho_n = \tilde{\kappa}_n \gamma_n \rho_{n-1}^2, \quad n > 1,
\]
where \(\gamma_n \in (0, 1]\) are known and decrease to zero faster than \(n^{-n}\).

For \(\bar{u} = (u_n)_{n=1}^\infty \in V^\infty\), let
\[
\|\bar{u}\|_\star = \sum_{n=1}^\infty \rho_n \|u_n\|_{H^1(\Omega)},
\]

Define \(V^\star = \{ \bar{u} \in V^\infty : \|\bar{u}\|_\star < \infty \}\), \(S_A^\star = S_A \cap V^\star\).
Clearly \(V^\star\) and \(S_A^\star\) are Banach spaces.
Main Results

We summarize our results in the commutative diagram

$$\begin{array}{ccc}
R & \xrightarrow{S(t)} & R \\
| & \downarrow{W(\cdot)} & \downarrow{W(\cdot)} \\
S^*_A & \xrightarrow{S_{\text{normal}}(t)} & S^*_A \\
| & \downarrow{Q(0,\cdot)} & \downarrow{Q(0,\cdot)} \\
V^* & \xrightarrow{S_{\text{ext}}(t)} & V^*
\end{array}$$

Figure: Commutative diagram

where all mappings are continuous.

(Foias-Hoang-Olson-Ziane) The normal form of the NSE

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Determining the weights

Recursive estimates: \(\rho_n \| W_n(u^0)\| \leq d_n\) where

\[
d_1 = \rho_1 \| u^0 \|, \quad d_n = \rho_n \| u^0 \| + \kappa_n g_0^n \left\{ X^2 + \left( \sum_{k=1}^{n-1} d_k \right)^2 \right\}, \quad n > 1
\]

where \(g_0, X\) are positive numbers depending on \(u^0\), \(\kappa_n\) can be chosen to be small.

Question: For which \(\kappa_n\) that \(\sum_{n=1}^{\infty} d_n\) is finite?

We find decreasing \(\rho_n\) such that \(\rho_n \leq \kappa_n \rho_{n-1}^2\)
Lemma

Let \((a_n)_{n=1}^{\infty}\) and \((k_n)_{n=2}^{\infty}\) be two sequences of positive numbers. Let \(d_1 = a_1\) and \(d_n = a_n + k_n(\sum_{k=1}^{n-1} d_k)^2\), for \(n > 1\). Suppose

\[
\lim_{n \to \infty} k_n^{1/2^n} = 0.
\]

If \(\sum_{n=1}^{\infty} a_n\) is finite, so is \(\sum_{n=1}^{\infty} d_n\). More precisely,

\[
\sum_{n=1}^{\infty} d_n \leq \sum_{n=1}^{\infty} a_n + \alpha^2 \sum_{n=1}^{\infty} k_n M^{2(2^n-1)} < \infty,
\]

where \(\alpha = \sup\{a_n : n \in \mathbb{N}\}\) and \(M = 3 \sup\{1, \alpha, k_n \alpha : n > 1\}\).

(Foias-Hoang-Olson-Ziane)
Recursive estimates

Sketch: Given $u^0 \in \mathcal{R}$, the asymptotic expansion of $u(t)$ is

$$u(t) \sim \sum u_n(t) = \sum W_n(t, u^0) e^{-nt} \quad \text{as} \quad t \to \infty.$$  

For $n \geq 2$, denote $\tilde{u}_n(t) = u(t) - \sum_{k=1}^{n-1} u_k(t)$.

Suppose we have estimates for $\xi_j = W_j(u^0)$, $q_j(\zeta) = W_j(\zeta, u^0)$ for $j = 1, \ldots, n-1$ and $\tilde{u}_j(\zeta)$ for $j = 2, \ldots, n$ for $\zeta$ in some domain of analyticity.

- Estimate $W_n(u^0) = \xi_n$ using

$$W_n(u^0) = R_n \tilde{u}_n(0) - \int_0^\infty e^{nt} \sum_{\substack{k,j \leq n-1 \\k+\j \geq n+1}} R_n B(u_k, u_j) d\tau$$

$$- \int_0^\infty e^{nt} R_n \left[ B(u, \tilde{u}_n) + B(\tilde{u}_n, u) - B(\tilde{u}_n, \tilde{u}_n) \right] d\tau.$$

for $n \in \sigma(A)$ and $n \geq 2$. 

(Foias-Hoang-olson-Ziane)
Estimate $q_n(0, \xi_1, \ldots, \xi_{n-1})$.

Using extended NSE with initial data $u_n(0)$ being the above $q_n(0)$ to bound $\rho_n \| W_n(\zeta, u^0) e^{-n\zeta} \| \leq M_n e^{-\text{Re}\zeta}$. Then use Fragmen-Linderlöf type esitimate to obtain exact rate of decay.

Using Navier–Stokes equations and Phragmen-Linderlöf type esitimate to bound bound $\| \tilde{u}_{n+1}(\zeta) \|$.

Above, we need to complexify NSE as well as extended NSE.
Formula of $q_n(0, \xi_1, \ldots, \xi_{n-1})$

Recall: $q_n(t)$ is the polynomial solution of

$$q'_n + (A - n)q_n + \beta_n = 0, \quad R_n q_n(0) = \xi_n,$$

$$\beta_n = \sum_{k+j=n} B(q_k, q_j).$$

Then

$$R_n q_n(0) = \xi_n$$

$$P_{n-1} q_n(0) = \int_0^\infty e^{\tau(A-n)} P_{n-1} P_{n-1} \beta_n(\tau) d\tau$$

$$(I - P_n) q_n(0) = - \int_{-\infty}^0 e^{\tau(A-n)} (P_{n^2} - P_n) (P_{n^2} - P_n) \beta_n(\tau) d\tau.$$
Extended Navier-Stokes Equations

Let \((\rho_n)_{n=1}^{\infty}\) be a sequence of positive numbers satisfying

\[
\rho_n = \kappa_n \min \{ \rho_k \rho_j : k + j = n \}, \quad \kappa_n \in (0, 1], \quad n \geq 2.
\]

with \(\lim_{n \to \infty} \kappa_n^{1/n} = 0\).

**Theorem**

*If \(\bar{u}^0 \in V^*\), then \(S_{\text{ext}}(t)\bar{u}^0 \in V^*\) for all \(t > 0\). More precisely,*

\[
\|S_{\text{ext}}(t)\bar{u}^0\|_* \leq Me^{-t}, \quad t > 0,
\]

*where \(M = \|\bar{u}^0\|_* + C_1 \sum_{n=2}^{\infty} \kappa_n (n - 1)M_0^n\), \(M_0 = \max\{1, 2C_1 \kappa_n (n - 1)\}\) \(\max\{1, 2\|\bar{u}^0\|_*\}\).*
Theorem

\( S_{\text{ext}}(t) \) is continuous from \( V^* \) to \( V^* \), for \( t \in [0, \infty) \). More precisely, for any \( \bar{u}^0 \in V^* \) and \( \varepsilon > 0 \), there is \( \delta > 0 \) such that

\[
\| S_{\text{ext}}(t)\bar{v}^0 - S_{\text{ext}}(t)\bar{u}^0 \|_* < \varepsilon e^{-t},
\]

for all \( \bar{v}^0 \in V^* \) satisfying \( \| \bar{v}^0 - \bar{u}^0 \|_* < \delta \) and for all \( t \geq 0 \).
Phragmen-Linderlöf type estimates.

**Theorem**

Let \( f(\zeta) \) be analytic on the right half plane \( H_0 \), bounded by a constant \( M \) and

\[
\sup_{x > 0} e^{\alpha x} |f(x)| < \infty,
\]

where \( \alpha \) is a positive number. Then

\[
|f(\zeta)| \leq M e^{-\alpha \Re \zeta}, \quad \zeta \in H_0.
\]

Our domain of analyticity when \( \|u^0\| \) is small

\[
D = \{ \tau + i\sigma : \tau > 0, |\sigma| < c \tau e^{\alpha \tau} \},
\]

where \( c, \alpha > 0 \).
Lemma

Let \( c \geq \sqrt{2}, \alpha > 0 \), then the transformation

\[
\phi(\zeta) = \zeta - \frac{1}{\alpha} \log(1 + \alpha \zeta)
\]

conformally maps \( D \) to a set containing the right half plane. Moreover, \( \phi([0, \infty)) = [0, \infty) \).

Corollary

Suppose \( u(\zeta) \) is analytic in \( D(c, \alpha) \) where \( c \geq \sqrt{2}, \alpha > 0 \),

\[
|u(\zeta)| \leq M, \quad \zeta \in D(c, \alpha),
\]

\[
\sup_{t>0} e^{nt} |u(t)| < \infty, \quad t > 0,
\]

where \( n \) is a positive constant. Then

\[
|u(\zeta)| \leq M e^{-n \Re \zeta} |1 + \alpha \zeta|^{n/\alpha}, \quad \zeta \in \phi^{-1}_\alpha(H_0).
\]
Corollary

Let \( q(\zeta) \) be a polynomial of degree less than or equal to \( p \) and

\[
|e^{-N\zeta} q(\zeta)| \leq M, \quad \zeta \in D.
\]

Then

\[
|q(\zeta)| \leq M|1 + \alpha \zeta|^{N/\alpha}, \quad \zeta \in \phi^{-1}(H_0),
\]

\[
|q(\zeta)| \leq M(p + 1)(1 + \alpha a + \alpha r_a)^{N/\alpha} \left(\frac{|\zeta| + a}{r_a}\right)^p, \quad \zeta \in \mathbb{C}.
\]
The range of the normalization map

Let $u^0 \in \mathcal{R}$, estimate $\| \mathcal{W}(u^0) \|_* = \sum_{n=1}^{\infty} \rho_n \| \mathcal{W}_n(u^0) \|_{H^1(\Omega)}$. 

Estimates when $\| u^0 \|$ is small.

Above estimates are adequate.

Estimates when $\| u^0 \|$ is large.

Combine above estimates on $[t_0, \infty)$, when $t_0$ is large, with the energy estimate on $[0, t_0)$. 

(Foias-Hoang-Olson-Ziane) 

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Let $u^0 \in \mathcal{R}$, estimate $\| W(u^0) \|^\star = \sum_{n=1}^\infty \rho_n \| W_n(u^0) \|_{H^1(\Omega)}$. 

Estimates when $\| u^0 \|$ is small. Above estimates are adequate.
The range of the normalization map

Let $u^0 \in \mathcal{R}$, estimate $\| \mathcal{W}(u^0) \|_* = \sum_{n=1}^{\infty} \rho_n \| \mathcal{W}_n(u^0) \|_{H^1(\Omega)}$.

**Estimates when $\| u^0 \|_*$ is small.** Above estimates are adequate.

**Estimates when $\| u^0 \|_*$ is large.** Combine above estimates on $[t_0, \infty)$, when $t_0$ is large, with the energy estimate on $[0, t_0)$.
Continuity of the normalization map, etc.

Similar to the estimates for the range. Final form: Given $u^0, v^0 \in R$, with $\|u^0 - v^0\| < 1$. Let $w^0 = u^0 - v^0$, $w(t) = u(t) - v(t)$. Then

$$\rho_n \|W_n(u^0) - W_n(v^0)\| \leq y_n,$$

$$y_1 = \rho_1 \|w^0\|,$$

$$y_n = \rho_n \|w^0\| + \kappa_n M^n \left( |w^0| + \|w(t_0)\| + \sum_{k=1}^{n-1} y_k \right),$$

where $M$ depends on $u^0$, positive $t_0$ is fixed.

**Lemma**

Given $\varepsilon > 0$, there is $\delta = \delta(u^0) > 0$ such that if $\|u^0 - v^0\| < \delta$, then $
\sum_{n=1}^{\infty} y_n < \varepsilon$. 

(Foias-Hoang-Olson-Ziane)
Summary

We have proved the commutative diagram

\[
\begin{array}{cccccc}
\mathcal{R} & \xrightarrow{S(t)} & \mathcal{R} \\
| & W(\cdot) & W(\cdot) & |
\end{array}
\]

\[
\begin{array}{cccccc}
W(0,\cdot) & \xrightarrow{S_A^*} & S_{\text{normal}}(t) & \xrightarrow{S_A^*} & W(0,\cdot) \\
Q(0,\cdot) & & S_{\text{ext}}(t) & & Q(0,\cdot) \\
V^* & \xrightarrow{S(t)} & V^*
\end{array}
\]

Figure: Commutative diagram

where all mappings are continuous.
Open problems

- Find $u^0$ such that $\sum_{n=1}^{\infty} \| W_n(0, u^0) \| < \varepsilon_0$ or $\limsup_{n \to \infty} \| W_n(0, u^0) \|^{1/n} < \infty$.
- Relations between the classical Leray weak solutions and the solutions to the extended Navier–Stokes equations.
- More properties and applications of the normalization map.