Navier-Stokes equations: the normalization map, statistical solutions and fluid dynamics

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Outline

1. Introduction
2. Properties of the normalization map
3. Asymptotic behavior of the mean flows
4. Beltrami flows
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Physical quantities in studies of Fluid Dynamics:

- the kinetic energy/mass

\[ \mathcal{E}(t) = \frac{1}{2} \int_{\Omega} |u(x, t)|^2 dx, \]

- the dissipation rate of energy/mass

\[ \mathcal{F}(t) = \int_{\Omega} |\omega(x, t)|^2 dx, \quad \omega = \nabla \times u. \]

- and the helicity/mass

\[ \mathcal{H}(t) = \int_{\Omega} u(x, t) \cdot \omega(x, t) dx. \]

Their ensemble averages \( \langle \mathcal{E}(t) \rangle, \langle \mathcal{F}(t) \rangle \) and \( \langle \mathcal{H}(t) \rangle \).
Turbulence!

**GOAL:** Use the analysis of Navier–Stokes equations to understand the above quantities.
Navier-Stokes equations (NSE) in $\mathbb{R}^3$ with a potential body force

\[
\begin{align*}
\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u &= -\nabla p - \nabla \phi, \\
\text{div } u &= 0, \\
u(x, 0) &= u^0(x),
\end{align*}
\]

$\nu > 0$ is the kinematic viscosity,

$u = (u_1, u_2, u_3)$ is the unknown velocity field,

$p \in \mathbb{R}$ is the unknown pressure,

$\phi$ is the potential of the body force,

$u^0$ is the initial velocity.
Let $L > 0$ and $\Omega = (0, L)^3$. The $L$-periodic solutions:

$$u(x + Le_j) = u(x) \text{ for all } x \in \mathbb{R}^3, j = 1, 2, 3,$$

where $\{e_1, e_2, e_3\}$ is the canonical basis in $\mathbb{R}^3$.

Zero average condition

$$\int_{\Omega} u(x) dx = 0,$$

Throughout $L = 2\pi$ and $\nu = 1$. 
Let $\mathcal{V}$ be the set of $\mathbb{R}^3$-valued $L$-periodic trigonometric polynomials which are divergence-free and satisfy the zero average condition.

We define

$$H = \text{closure of } \mathcal{V} \text{ in } L^2(\Omega)^3 = H^0(\Omega)^3,$$

$$\mathcal{V} = \text{closure of } \mathcal{V} \text{ in } H^1(\Omega)^3,$$

$$\mathcal{D}_A = \mathcal{V} \cap H^2(\Omega)^3.$$

Norm on $H$: $|u| = \|u\|_H = \|u\|_{L^2(\Omega)}$,
Norm on $\mathcal{V}$: $\|u\| = \|u\|_V = \|\nabla u\|_{L^2(\Omega)} = \|\nabla \times u\|_{L^2(\Omega)}$,
Norm on $\mathcal{D}_A$: $\|\Delta u\|_{L^2(\Omega)}$.

The Stokes operator: $Au = -\Delta u$ for all $u \in \mathcal{D}_A$.

Spectrum of $A$: $\sigma(A) = \{|k|^2, 0 \neq k \in \mathbb{Z}^3\}$.

If $N \in \sigma(A)$, denote by $R_NH$ the eigenspace of $A$ corresponding to $N$.
Otherwise, $R_NH = \{0\}$. 


The bilinear mapping:

$$B(u, v) = P_L(u \cdot \nabla v) \text{ for all } u, v \in D_A.$$ 

$P_L$ is the Leray projection from $L^2(\Omega)$ onto $H$.

The functional form of the NSE:

$$\frac{du(t)}{dt} + Au(t) + B(u(t), u(t)) = 0, \quad t > 0,$$

$$u(0) = u^0.$$
Recall that for each \( u_0 \in H \), there exists a Leray-Hopf weak solution \( u(t) \) of the Navier–Stokes equations with \( u(0) = u_0 \). This weak solution satisfies

\[
    u \in C([0, \infty), H_{\text{weak}}) \cap L^\infty((0, \infty), H) \cap L^2((0, \infty), V).
\]

Additionally, let \( G = G(u(\cdot)) \) be the set of \( t_0 \geq 0 \) such that

\[
    \lim_{\tau \downarrow 0} \| u(t_0 + \tau) - u(t_0) \|_H = 0,
\]

then \( 0 \in G \), the Lebesgue measure of \( [0, \infty) \setminus G \) is zero and for any \( t_0 \in G \)

\[
    \| u(t) \|_H^2 + 2 \int_{t_0}^{t} \| u(s) \|_V^2 \, ds \leq \| u(t_0) \|_H^2, \quad t \geq t_0.
\]

Denote by \( \Sigma \) the set of the Leray-Hopf weak solutions of the Navier–Stokes equations on \([0, \infty)\). Hence \( \Sigma \subset C([0, \infty), H_{\text{weak}}) \).
We denote by $T$ the class of test functionals

$$
\Phi(u) = \phi(\langle u, g_1 \rangle, \langle u, g_2 \rangle, \ldots, \langle u, g_k \rangle), \quad u \in H,
$$

for some $k > 0$, where $\phi$ is a $C^1$ function on $\mathbb{R}^k$ with compact support and $g_1, g_2, \ldots, g_k$ are in $V$.

A family $\{\mu_t, t \geq 0\}$ of Borel probability measures on $H$ is called a statistical solution of the Navier–Stokes equations with the initial data $\mu_0$ if

- the initial kinetic energy $\int_H \|u\|^2_H d\mu_0(u)$ is finite;
- the function $t \mapsto \int_H \varphi(u) d\mu_t(u)$ is measurable for every bounded and continuous function $\varphi$ on $H$;
- the function $t \mapsto \int_H \|u\|^2_H d\mu_t(u)$ belongs to $L^\infty_{loc}([0, \infty))$;
the function \( t \mapsto \int_H \| u \|_V^2 d\mu_t(u) \) belongs to \( L^1_{loc}([0, \infty)) \);

- \( \mu_t \) satisfies the Liouville equation

\[
\int_H \Phi(u) d\mu_t(u) = \int_H \Phi(u) d\mu_0(u) \\
- \int_0^t \int_H \langle Au + B(u, u), \Phi'(u) \rangle d\mu_s(u) ds,
\]

for all \( t \geq 0 \) and \( \Phi \in \mathcal{T} \);

- the following energy inequality holds

\[
\int_H \| u \|_H^2 d\mu_t(u) + 2 \int_0^t \int_H \| u \|_V^2 d\mu_s(u) ds \leq \int_H \| u \|_H^2 d\mu_0(u).
\]
A statistical solution \( \{ \mu_t, t \geq 0 \} \) is called a Vishik-Fursikov (VF) statistical solution if there is a Borel probability measure \( \hat{\mu} \), called the Vishik-Fursikov (VF) measure, on the space \( C([0, \infty), H_{\text{weak}}) \), such that

- \( \hat{\mu}(\Sigma) = 1 \);
- for each \( t \geq 0 \), \( \mu_t \) is the projection measure \( Pr_t \hat{\mu} \) on \( H \), i.e.

\[
\int_H \Phi(u) d\mu_t(u) = \int_\Sigma \Phi(v(t)) d\hat{\mu}(v(\cdot)),
\]

for all \( \Phi \in C(H_{\text{weak}}) \).
The existence of the statistical solutions

We summarize the results by Foias and Vishik-Fursikov.

**Theorem**

Let $m$ be a Borel probability measure on $H$ such that $\int_H \|u\|^2_H dm(u)$ is finite. Then there exists a Vishik-Fursikov statistical solution $\{\mu_t, t \geq 0\}$ with $\mu_0 = m$.

**Note:** The Vishik-Fursikov statistical solutions and measures are not necessarily unique.
Denote by $\mathcal{R}$ the set of all initial data $u^0 \in \mathcal{V}$ such that the solution is regular for all times $t > 0$.

For $u^0 \in \mathcal{R}$, the solution $u(t) = u(t, u^0)$ has the asymptotic expansion of

$$u(t) \sim q_1(t)e^{-t} + q_2(t)e^{-2t} + q_3(t)e^{-3t} + \ldots,$$

where $q_j(t) = W_j(t, u^0)$ is a polynomial in $t$ of degree at most $(j - 1)$ and with values in $\mathcal{V}$.

One has for $N \in \mathbb{N}$ and $m \in \mathbb{N}$ that

$$\|u(t) - \sum_{j=1}^{N} q_j(t)e^{-jt}\|_{H^m(\Omega)} = O(e^{-(N+\varepsilon)t})$$

as $t \to \infty$, for some $\varepsilon = \varepsilon_{N,m} > 0$. 
Normalization map

Let

\[ W(u^0) = \left( W_1(u^0), W_2(u^0), W_3(u^0), \ldots \right), \]

where \( W_j(u^0) = R_j q_j(0) \), for \( j = 1, 2, 3, \ldots \).

Here we focus on the first component

\[ W_1(u^0) = \lim_{t \to \infty} e^t u(t) = \lim_{t \to \infty} e^t R_1 u(t). \]

Lemma

Let \( u(\cdot) \in \Sigma \) and \( t_0 \geq 0 \) such that \( u(t_0) \in \mathcal{R} \). Then

\[ e^t W_1(u(t)) = e^{t_0} W_1(u(t_0)), \quad t \geq t_0. \]
Properties of the normalization map

**Definition**

Let $u(\cdot) \in \Sigma$. We define

$$W_1(u(\cdot)) = e^{t_0} W_1(u(t_0)),$$

where $t_0 \geq 0$ such that $u(t_0) \in \mathcal{R}$.

One also has

$$W_1(u(\cdot)) = \lim_{t \to \infty} e^t u(t) = \lim_{t \to \infty} e^t R_1 u(t).$$

where the limits are taken in either $H$ or $V$.

**Note:** If $u_0 = u(0) \in \mathcal{R}$, then $t_0 = 0$ and $W_1(u(\cdot)) = W_1(u_0)$. Thus $W_1(u(\cdot))$ is an extension of $W_1(u_0), u_0 \in \mathcal{R}$. 
Properties of the normalization map (cont.)

Let \( u(\cdot) \in \Sigma \). Then

\[
\| W_1(u(\cdot)) \|_H \leq e^t \| u(t) \|_H, \quad t \in \mathcal{G}(u(\cdot)).
\]

In particular,

\[
\| W_1(u(\cdot)) \|_H \leq \| u(0) \|_H.
\]

For \( t \geq 0 \),

\[
\| W_1(u(\cdot)) \|_H^2 \leq \frac{1}{T} \int_t^{t+T} e^{2\tau} \| u(\tau) \|_H^2 d\tau \leq \| u(0) \|_H^2,
\]

\[
\int_t^{t+T} \| u(\tau) \|_V^2 d\tau \leq \frac{e^{-2t}}{2} \| u(0) \|_H^2 - \frac{e^{-2(t+T)}}{2} \| W_1(u(\cdot)) \|_H^2.
\]

\[
\| R_1 u(0) - W_1(u(\cdot)) \|_H \leq C_1 \| u(0) \|_H^2.
\]
Asymptotic behavior of the mean flows.

**Proposition**

\[
\lim_{t \to \infty} e^{2t} \int_{H} \| u \|_{H}^2 d\mu_t(u) = \int_{\Sigma} \| W_1(u(\cdot)) \|_{H}^2 d\hat{\mu}(u(\cdot)).
\]

**Theorem**

We have for any \( T > 0 \) that

\[
\lim_{t \to \infty} \frac{e^{2t}}{T} \int_{t}^{t+T} \int_{H} \| u \|_{H}^2 d\mu_s(u) ds
\]

\[
= 1 - e^{-2T} \int_{\Sigma} \| W_1(u(\cdot)) \|_{H}^2 d\hat{\mu}(u(\cdot)),
\]
Theorem (continued)

\[ \lim_{t \to \infty} \frac{e^{2t}}{T} \int_t^{t+T} \int_H \|u\|_V^2 d\mu_s(u) ds = \frac{1 - e^{-2T}}{2T} \int_{\Sigma} \|W_1(u(\cdot))\|_H^2 d\hat{\mu}(u(\cdot)), \]

and

\[ \lim_{t \to \infty} \frac{e^{2t}}{T} \int_t^{t+T} \int_H \mathcal{H}(u) d\mu_s(u) ds = \frac{1 - e^{-2T}}{2T} \int_{\Sigma} \mathcal{H}(W_1(u(\cdot))) d\hat{\mu}(u(\cdot)). \]
Proposition

For any $T > 0$ and $t \geq 0$,

$$e^{-2(t+T)} \int_{\Sigma} \|W_1 u(\cdot)\|_{H}^2 d\hat{\mu}(u(\cdot)) \leq \frac{1}{T} \int_{t}^{t+T} \int_{H} \|u\|_{H}^2 d\mu_\tau(u) d\tau$$

$$\leq e^{-2t} \int_{H} \|u\|_{H}^2 d\mu_0(u),$$

and

$$e^{-2(t+T)} \int_{\Sigma} \|W_1 u(\cdot)\|_{H}^2 d\hat{\mu}(u(\cdot)) \leq \frac{1}{T} \int_{t}^{t+T} \int_{H} \|u\|_{H}^2 d\mu_\tau(u) d\tau$$

$$\leq \frac{e^{-2t}}{2T} \int_{H} \|u\|_{H}^2 d\mu_0(u) - \frac{e^{-2(t+T)}}{2T} \int_{\Sigma} \|W_1(u(\cdot))\|_{H}^2 d\hat{\mu}(u(\cdot)).$$
A $C^1$ vector field $u(x)$ in $\mathbb{R}^3$ is said to be Beltrami if
\[
\nabla \times u(x) = \alpha(x)u(x), \quad x \in \mathbb{R}^3, \text{ some } \alpha(x) \in \mathbb{R}.
\]

If $u = u(\cdot)$ is an eigenfunction of the curl operator $\mathcal{C}$, then it is Beltrami with $\alpha \equiv \pm \sqrt{n}$, for some $n \in \sigma(A)$.

**Proposition**

Let $u \in R_nH \setminus \{0\}$ where $n \in \sigma(A)$. Then $u$ is Beltrami if and only if $u$ is an eigenfunction of the curl operator, i.e., $\mathcal{C}u = \sqrt{n}u$ or $\mathcal{C}u = -\sqrt{n}u$.

**Notation:** $R_n^{\pm}H$ is the eigenspace of the curl operator corresponding to $\pm \sqrt{n}$.
Asymptotic Beltrami flows

**Definition**

We say that a time dependent vector field \( u(x, t) \) is asymptotically Beltrami if there are \( \alpha(x, t) \in \mathbb{R} \) such that

\[
\lim_{t \to \infty} \frac{\nabla \times u(x, t) - \alpha(x, t)u(x, t)}{|u(x, t)|} = 0, \text{ a.e. on } \mathbb{R}^3.
\]

Let \( u(\cdot) \in \Sigma \) such that there is \( t_0 \geq 0, u(t_0) \in \mathcal{R} \setminus \{0\} \). Denote

\[
n_\ast = n_\ast (u(\cdot)) = \lim_{t \to \infty} \frac{\|u(t)\|_V^2}{\|u(t)\|_H^2}.
\]

Define

\[
W_\ast (u(\cdot)) = e^{n_\ast t_0} W_{n_\ast} (u(t_0)) = \lim_{t \to \infty} e^{n_\ast t} u(t),
\]

where the limit is taken in either \( H \) or \( V \).
Theorem

Let \( u(\cdot) \in \Sigma \) such that \( u(t_0) \in \mathcal{R} \setminus \{0\} \), for some \( t_0 > 0 \). The following are equivalent

1. \( u(t) \) is asymptotically Beltrami.
2. There is a subsequence \( t_k \uparrow \infty \) and \( \alpha(x, t_k) \in \mathbb{R} \) such that
   \[
   \lim_{k \to \infty} \frac{\nabla \times u(x, t_k) - \alpha(x, t_k)u(x, t_k)}{|u(x, t_k)|} = 0,
   \]
a.e. on \( \mathbb{R}^3 \).
3. \( W_*(u(\cdot)) \) is a Beltrami vector field.
4. For \( n_* = n_*(u(\cdot)) \),
   \[
   \lim_{t \to \infty} \frac{\|\mathcal{C}u(t) - \varepsilon \sqrt{n_*}u(t)\|_{L^2(\Omega)}}{\|u(t)\|_{L^2(\Omega)}} = 0,
   \]
   where \( \varepsilon = 1 \) or \(-1\).
Definition

Let $\hat{\mu}$ be a Vishik-Fursikov measure on $\Sigma$. We say that the $\hat{\mu}$ is asymptotically Beltrami if almost surely every solution $u(\cdot)$ in $\Sigma$ is asymptotic Beltrami; more precisely,

$$\hat{\mu}\left(\{u(\cdot) \in \Sigma : u(\cdot) \text{ is asymptotically Beltrami}\}\right) = 1.$$ 

The necessary condition for $\hat{\mu}$ to be asymptotically Beltrami is that

$$\hat{\mu}\left(\{u(\cdot) \in \Sigma : |R_1^+ W_1(u(\cdot))| |R_1^- W_1(u(\cdot))| = 0\}\right) = 1,$$

equivalently,

$$\int_{\Sigma} |R_1^+ W_1(u(\cdot))| |R_1^- W_1(u(\cdot))| d\hat{\mu}(u(\cdot)) = 0,$$

or

$$\int_{\Sigma} |R_1^+ W_1(u(\cdot))| + |R_1^- W_1(u(\cdot))| - |W_1(u(\cdot))| d\hat{\mu}(u(\cdot)) = 0.$$
Proposition

If \( \hat{\mu} \) is a VF measure with initial data \( \mu \) satisfying

\[
\int_H \left[ |R_1^+ u| + |R_1^- u| - |R_1 u| \right] d\mu(u) > 3c_5 \int_H |u|^2 d\mu(u),
\]

then \( \hat{\mu} \) is not asymptotically Beltrami.

Theorem

There exists a VF measure \( \hat{\mu} \) with initial Gaussian probability measure such that \( \hat{\mu} \) is not asymptotically Beltrami.
Fast decaying mean flows

We study the case when

\[ \int_{\Sigma} \| W_1(u(\cdot)) \|^2_H d\hat{\mu}(u(\cdot)) = 0. \]

Denote

\[ \mathcal{M}_1 = \{ u \in \mathcal{R} : W_1(u) = 0 \}. \]

Let

\[ \mathcal{N}_1 = \{ u_0 \in H : \exists u(\cdot) \in \Sigma, u(0) = u_0, u(t) \in \mathcal{M}_1, t \geq t_0(u(\cdot)) \}, \]

\[ \Sigma_1 = \{ u(\cdot) \in \Sigma : u(t) \in \mathcal{M}_1, t \geq t_0, \text{ some } t_0 \text{ depending on } u(\cdot) \}. \]
**Proposition**

\[
\int_{\Sigma} \| W_1(u(\cdot)) \|^2_H d\hat{\mu}(u(\cdot)) = 0 \text{ if and only if } \hat{\mu}(\Sigma_1) = 1,
\]
equivalently,

\[
\int_{\Sigma} \| W_1(u(\cdot)) \|^2_H d\hat{\mu}(u(\cdot)) > 0 \text{ if and only if } \hat{\mu}(\Sigma_1) < 1,
\]

**Corollary**

If \( \int_{\Sigma} \| W_1(u(\cdot)) \|^2_H d\hat{\mu}(u(\cdot)) = 0 \) then \( \mu_0(\mathcal{N}_1) = 1 \).
Generic properties

Let $\hat{\mu}$ and $\tilde{\mu}$ be two Borel measures on $\Sigma$. We define $d_1(\hat{\mu}, \tilde{\mu})$ by the total variation of the measure $\hat{\mu} - \tilde{\mu}$, that is,

$$d_1(\hat{\mu}, \tilde{\mu}) = \sup \left\{ \sum_{j=1}^{N} |\hat{\mu}(E_j) - \tilde{\mu}(E_j)| \right\},$$

where the supremum is taken over all Borel partitions $\{E_1, E_2, \ldots, E_N\}$, $N \in \mathbb{N}$, of $\Sigma$.

Let $\mathcal{M}$ be the set of all VF measures and define the following metric in $\mathcal{M}$:

$$d(\hat{\mu}, \tilde{\mu}) = d_1(\hat{\mu}, \tilde{\mu}) + \int_{\Sigma} |u(0)|^2 d|\hat{\mu} - \tilde{\mu}|(u(\cdot)),$$

where $|\hat{\mu} - \tilde{\mu}|$ is the total variation measure of the signed measure $(\hat{\mu} - \tilde{\mu})$.

Proposition

*The metric space $(\mathcal{M}, d)$ is complete.*
A property $P(\hat{\mu})$ of a VF measure $\hat{\mu}$ is called \textit{generic} if the set of all VF measures $\hat{\mu}$ enjoying the property $P(\hat{\mu})$ contains an intersection of dense open sets in $\mathcal{M}$.

**Theorem**

The set $\mathcal{M}_E$ of all $\hat{\mu} \in \mathcal{M}$ such that

$$\int_\Sigma |W_1(u(\cdot))|^2 d\hat{\mu}(u(\cdot)) > 0$$

(1)

holds is open and dense in $\mathcal{M}$. Subsequently, (1) is generic.

Denote by $\mathcal{M}_H$ the set of all $\hat{\mu} \in \mathcal{M}$ such that

$$\int_\Sigma H(W_1(u(\cdot))) d\hat{\mu}(u(\cdot)) \neq 0.$$

(2)

holds.

**Theorem**

The set $\mathcal{M}_H$ is open and dense in $\mathcal{M}$. Subsequently, (2) is generic.
The genericity of the VF measures which are asymptotically Beltrami: let

\[ M_B = \{ \hat{\mu} \in \mathcal{M} : \hat{\mu} \text{ is asymptotically Beltrami} \}, \]

\[ \mathcal{N}_B = \left\{ \hat{\mu} \in \mathcal{M} : \int_{\Sigma} |R_1^+ W_1(u(\cdot))| \ |R_1^- W_1(u(\cdot))| d\hat{\mu}(u(\cdot)) > 0 \right\}. \]

Note: \( \mathcal{N}_B \subset \mathcal{M} \setminus M_B \).

**Theorem**

The set \( \mathcal{N}_B \) is open and dense in \( \mathcal{M} \). Consequently, the property “\( \hat{\mu} \) is not asymptotically Beltrami” for a VF measure \( \hat{\mu} \) is generic.

**Idea**

Let \( \hat{\mu}, \hat{m} \in \mathcal{M}, \ \varepsilon \in (0, 1) \) and \( \check{\mu} = (1 - \varepsilon)\hat{\mu} + \varepsilon \hat{m} \). Then \( \check{\mu} \in \mathcal{M} \) and

\[ d(\check{\mu}, \hat{\mu}) \leq 2\varepsilon + \varepsilon \left\{ \int_{\Sigma} |u(0)|^2 d\hat{\mu}(u(\cdot)) + \int_{\Sigma} |u(0)|^2 d\hat{m}(u(\cdot)) \right\}. \]
Kolmogorov’s empirical theory of turbulence: let

\[ U^2 = \frac{1}{L^3} \langle \int_{[0,L]^3} |\mathbf{u}(x, t)|^2 \, dx \rangle \quad \text{and} \quad \epsilon = \frac{\nu}{L^3} \langle \int_{[0,L]^3} |\nabla \times \mathbf{u}(x, t)|^2 \, dx \rangle, \]

where \( \langle \cdot \rangle \) denotes an ensemble average. These two quantities are connected by

\[ U^2 \sim \int_{k_i}^{k_d} S(k) \, dk, \quad \epsilon \sim \nu \int_{k_i}^{k_d} k^2 S(k) \, dk, \]

where \( S(k) \) is the energy spectrum and \([k_i, k_d]\) is so called the “inertial range” of the turbulent flows. Assume \( k_i \sim k_0 = \sqrt{\lambda_1} = 2\pi / L, \) \( k_d \sim (\epsilon / \nu^3)^{1/4} \) and \( S(k) \sim \epsilon^{2/3} k^{-5/3} \) (based on the dimensional analysis), we obtain

\[ U^2 \sim \epsilon^{2/3} \int_{k_i}^{k_d} k^{-5/3} \, dk \sim \epsilon^{2/3} k_i^{-2/3} \sim (L\epsilon)^{2/3}. \]
Let \((\mu_t)_{t \geq 0}\) be a VF statistical solution to the Navier–Stokes equations with the VF measure \(\hat{\mu}\) and \(T > 0\). We define for \(t \geq 0\)

\[
U_t^2 = \lambda_1^{3/2} \frac{1}{T} \int_t^{t+T} \int_H |u|^2 d\mu_\tau(u) d\tau,
\]

and

\[
\epsilon_t = \nu \lambda_1^{3/2} \frac{1}{T} \int_t^{t+T} \int_H \|u\|^2 d\mu_\tau(u) d\tau,
\]

The first component of the normalization map is defined now by

\[
W_1(u(\cdot)) = \lim_{t \to \infty} e^{\nu \lambda_1 t} u(t),
\]

where the limit is taken in any Sobolev norms.
Let

\[ \alpha_0^2 = \lambda_1^{3/2} \int_H |u|^2 d\mu_0(u) \text{ and } \alpha_1^2 = \lambda_1^{3/2} \int_{\Sigma} |W_1(u(\cdot))|^2 d\hat{\mu}(u(\cdot)). \]

**Proposition**

One has for each \( T > 0 \) that

\[
\lim_{t \to \infty} e^{2\nu \lambda_1 t} U_t^2 = \frac{1 - e^{-2T}}{2T} \alpha_1^2,
\]

\[
\lim_{t \to \infty} e^{2\nu \lambda_1 t} \epsilon_t = \frac{1 - e^{-2T}}{2T} \alpha_1^2.
\]

If Kolmogorov’s theory applies to \( U_t^2 \) and \( \epsilon_t \) then there are absolute positive constants \( c_K \) and \( C_K \) such that

\[
c_K \leq \frac{U_t^2}{(L/2\pi)^2/3 \epsilon_t^{2/3}} = \frac{\lambda_1^{1/3} U_t^2}{\epsilon_t^{2/3}} \leq C_K.
\]
Proposition

For $T > 0$ and $t \geq 0$, one has

\[ e^{-2\nu\lambda_1(t+T)}\alpha_1^2 \leq U_t^2 \leq e^{-2\nu\lambda_1 t}\alpha_0^2, \]

\[ \nu\lambda_1 U_t^2 \leq \epsilon_t \leq \frac{e^{-2\nu\lambda_1 t}}{2T}(\alpha_0^2 - e^{-2\nu\lambda_1 T}\alpha_1^2). \]

Consequently, let $Q = \frac{\alpha_1^2}{\alpha_0^2}$, one has for $t \geq 0$ that

\[ \left\{ \frac{2\nu\lambda_1 T}{Q^{-1} e^{2\nu\lambda_1 T} - 1} \right\}^{2/3} \left\{ \frac{e^{-2\nu\lambda_1 t}\alpha_1^2}{\lambda_1 \nu^2} \right\}^{1/3} \leq \frac{\lambda_1^{1/3} U_t^2}{\epsilon_t^{2/3}} \leq \left\{ \frac{e^{-2\nu\lambda_1 t}\alpha_0^2}{\lambda_1 \nu^2} \right\}^{1/3}. \]
**Corollary**

Kolmogorov’s universal features may only be valid on the time interval \([t_K, T_K]\) where

\[
t_K = \frac{1}{2\nu \lambda_1} \left( \log \frac{\alpha_0^2}{\lambda_1 \nu^2} - 3 \log C_K - 2 \log \frac{Q^{-1} e^{2\nu \lambda_1 T}}{2\nu \lambda_1 T} - 1 \right),
\]

\[
T_K = \frac{1}{2\nu \lambda_1} \left( \log \frac{\alpha_0^2}{\lambda_1 \nu^2} - 3 \log c_K \right).
\]

**Example**

Let \(L = 2\pi\) (\(\lambda_1 = 1\)), \(\nu = 1\), \(M > 0\), and \(\theta \in (0, 1)\). There is a VF measure \(\hat{\mu}\) with initial Gaussian data such that \(\alpha_0^2 \geq M\) and \(\theta \leq Q \leq 1\), hence one obtains

\[
t_K \geq \frac{1}{2} \left( \log M - 3 \log C_K - 2 \log \frac{\theta^{-1} e^{2T}}{2T} - 1 \right),
\]

\[
T_K = \frac{1}{2} \{\log(\alpha_0^2) - 3 \log c_K\}.
\]