On the Stokes and Laplacian operators in Navier–Stokes equations

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Introduction: NSE with Navier boundary condition

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Navier boundary condtion:

$$u \cdot N = 0, \quad \nu[(Du)N]_{\mathrm{tan}} = 0, \quad \text{on } \partial\Omega,$$

where N is the unit outward normal vector to the boundary, $[\cdot]_{\mathrm{tan}}$ means the tangential part.

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Assume $\nu = 1$.

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Can we improve this?

Setting in General Domain

Consider an open, bounded, connected domain $\Omega \subset \mathbb{R}^3$ with C^3 boundary satisfying:
Condition (B). For $a, b \in \mathbb{R}^3$, if $(a + b \times x) \cdot N = 0$ on $\partial\Omega$, then $b = 0$.

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Let $\tilde{H} = \{u \in L^2(\Omega, \mathbb{R}^3) : \nabla \cdot u = 0 \text{ in } \Omega \text{ and } u \cdot N = 0 \text{ on } \partial\Omega\}$.
We have the Helmholtz-Leray decomposition

$$L^2(\Omega, \mathbb{R}^3) = \tilde{H} \oplus \tilde{H}^\perp \text{ where } \tilde{H}^\perp = \{\nabla\phi : \phi \in H^1(\Omega)\}.$$

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The subspace H_0 arises from the variational formulation of Navier–Stokes equations with Navier boundary condtion.

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Condition (B) implies $H_0 = \{0\}$ and $P = \tilde{P}$.

Setting in Thin Domain

Consider three dimensional thin domains of the form

$$\Omega'_\varepsilon = \{(x_1, x_2, x_3) : (x_1, x_2) \in \mathbb{R}^2, \varepsilon g_0(x_1, x_2) < x_3 < \varepsilon g_1(x_1, x_2)\},$$

where $\varepsilon \in (0, 1]$, g_0 and g_1 are given C^3 scalar functions in $\mathbb{R}^2$ satisfying

$$g_i(x' + \mathbf{e}_j) = g_i(x'), \quad x' \in \mathbb{R}^2, \ i = 0, 1, \ j = 1, 2,$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the standard basis of $\mathbb{R}^3$. We assume that

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We study the divergence-free vector fields $u(x)$ in Ω'_ε that satisfy the periodicity condition

$$u(x + \mathbf{e}_j) = u(x) \quad \text{for all} \quad x \in \Omega'_\varepsilon, \quad j = 1, 2,$$

and the Navier boundary condition on Γ'.

Let $L^2_{\mathrm{per}}(\Omega'_\varepsilon)$, resp. $H^k_{\mathrm{per}}(\Omega'_\varepsilon), k \geq 1$, be the closure with respect to the norm $\|\cdot\|_{L^2(\Omega_\varepsilon)}$, resp. $\|\cdot\|_{H^k(\Omega_\varepsilon)}$, of the set of all functions $\varphi \in C^\infty(\overline{\Omega'_\varepsilon})$ satisfying

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We define

$$H_0 = \{u = a + b \times x \in L^2_{\mathrm{per}}(\Omega'_\varepsilon, \mathbb{R}^3), u \cdot N = 0 \text{ on } \Gamma',\ \text{where } a, b \in \mathbb{R}^3\}.$$

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Let $u = a + b \times x \in H_0$, where $a, b \in \mathbb{R}^3$. The periodicity condition implise $b = 0$. Hence Condition (B) is satisfied and therefore $H_0 = \{0\}$.
The functional spaces and the Stokes operator are defined as usual (with the periodicity condition).

Main reults

Theorem (General Domain)

Let $u \in D_A$*, then*

$$\|Au + \Delta u\|_{L^2(\Omega)} \leq C\|u\|_{H^1(\Omega)},$$

where C *is a positive constant depending on the domain.*

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Theorem (Thin Domain)

Let $u \in D_A$*, then*

$$\|Au + \Delta u\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon\|\nabla u\|_{L^2(\Omega_\varepsilon)} + C\|u\|_{L^2(\Omega_\varepsilon)},$$

where the positive constant C *is independent of* ε*.*

Proofs of Main Results

Lemma (V. Busuioc – T. S. Ratiu)

Let $\mathcal{O}$ be an open subset of $\mathbb{R}^3$ such that $\Gamma_* = \partial\Omega \cap \mathcal{O} \neq \emptyset$. Let u belong to $C^1(\overline{\Omega} \cap \mathcal{O}, \mathbb{R}^3)$ and satisfy Navier boundary condition on Γ_*. Suppose $\check{N} \in C^1(\overline{\Omega} \cap \mathcal{O}, \mathbb{R}^3)$ with the restriction $\check{N}\big|_{\Gamma_*}$ being a unit normal vector field on Γ_*. Then

$$\check{N} \times (\nabla \times u) = 2\check{N} \times (\check{N} \times ((\nabla \check{N})^* u)) \quad on \quad \Gamma_*.$$

Proof.

Let $\omega = \nabla \times u$. From the identity $\check{N} \times \nabla(u \cdot \check{N}) = 0$ on Γ_*, we have

$$\begin{aligned} 0 &= \check{N} \times [(\nabla u)^* \check{N}] + \check{N} \times [(\nabla \check{N})^* u] \\ &= \check{N} \times [(Du)\check{N} - (Ku)\check{N}] + \check{N} \times [(\nabla \check{N})^* u], \end{aligned}$$

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Since $(Du)\check{N}$ is co-linear to $\check{N}$, we thus have

$$\check{N} \times [(\nabla \check{N})^* u] = \check{N} \times [(Ku)\check{N}] = \check{N} \times [(1/2)\omega \times \check{N}].$$

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Therefore $\check{N} \times (\omega \times \check{N}) = 2\check{N} \times [(\nabla \check{N})^* u]$.
Then use the identity

$$a \times (a \times (a \times b))) = -|a|^2(a \times b).$$

$\square$

Lemma (Key Lemma)

Let $u \in D_A$ *and* $\Phi \in H^{\perp}$. *Then*

$$\left| \int_{\Omega} \Delta u \cdot \Phi dx \right| \leq C \|\Phi\|_{L^2(\Omega)} \|u\|_{H^1(\Omega)},$$

where $C > 0$ *depends on* Ω.

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$$\begin{aligned}
\int_\Omega \Delta u \cdot \Phi dx &= -\int_\Omega (\nabla \times \omega) \cdot \Phi dx \\
&= -\int_\Omega \omega \cdot (\nabla \times \Phi) dx - \int_{\partial\Omega} (\omega \times \Phi) \cdot N d\sigma \\
&= \int_{\partial\Omega} (\omega \times N) \cdot \Phi d\sigma.
\end{aligned}$$

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Let $N(x), x \in \Omega$, be a C^2-extension of N.
Define $G(u) = N \times [(\nabla N)^* u]$ on $\overline{\Omega}$.

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Define $G(u) = N \times [(\nabla N)^* u]$ on $\overline{\Omega}$.
By Busuioc-Ratiu Lemma, we have

$$2N \times G(u)\big|_{\partial\Omega} = N \times \omega.$$

□

(continued).

We thus have

$$\begin{aligned}
\int_{\Omega} \Delta u \cdot \Phi dx &= -\int_{\partial\Omega} 2(N \times G(u)) \cdot \Phi d\sigma \\
&= \int_{\partial\Omega} 2(\Phi \times G(u)) \cdot N d\sigma \\
&= 2\int_{\Omega} \nabla \cdot (\Phi \times G(u)) dx \\
&= 2\int_{\Omega} \Phi \cdot (\nabla \times G(u)) - (\nabla \times \Phi) \cdot G(u) dx \\
&= 2\int_{\Omega} \Phi \cdot (\nabla \times G(u)) dx.
\end{aligned}$$

(continued).

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&= 2\int_{\Omega} \nabla \cdot (\Phi \times G(u)) dx \\
&= 2\int_{\Omega} \Phi \cdot (\nabla \times G(u)) - (\nabla \times \Phi) \cdot G(u) dx \\
&= 2\int_{\Omega} \Phi \cdot (\nabla \times G(u)) dx.
\end{aligned}
$$

Since $|\nabla \times G(u)| \leq C(|\nabla u| + |u|)$, we obtain

$$
\left| \int_{\Omega} \Delta u \cdot \Phi dx \right| \leq C \int_{\Omega} |\Phi|(|\nabla u| + |u|) dx \leq C\|\Phi\|_{L^2}\|u\|_{H^1}.
$$

Proof of the Inequality in General Domain.

Let $\Phi = Au + \Delta u = -P\Delta u + \Delta u$, then $\Phi \in H^{\perp}$. Since Au and Φ are orthogonal in $L^2(\Omega, \mathbb{R}^3)$, we have

$$\int_\Omega |\Phi|^2 dx = \int_\Omega (Au + \Delta u) \cdot \Phi dx = \int_\Omega \Delta u \cdot \Phi dx.$$

Applying the Key Lemma, we obtain

$$\|\Phi\|_{L^2}^2 \leq C \|\Phi\|_{L^2} \|u\|_{H^1}.$$

$\square$

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Then we have

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In the proofs above the role of condition (B) is to obtain $H_0 = \{0\}$. In fact, we have proved, without using condition (B), the following estimate:

Theorem

Even when Condition (B) is not satisfied, we have

$$\|\tilde{P}\Delta u - \Delta u\|_{L^2(\Omega)} \leq C\|u\|_{H^1(\Omega)},$$

for $u \in H^2(\Omega, \mathbb{R}^3) \cap \tilde{H}$ satisfying the Navier boundary condition on $\partial\Omega$.

Spherical Domain

Consider the following spherical domains

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Theorem

Let $R' > R > 0$ and $u \in H^2(\Omega_{R,R'}, \mathbb{R}^3) \cap \tilde{H}$ satisfying the Navier boundary condition on $\partial\Omega_{R,R'}$, then

$$\|\tilde{P}\Delta u - \Delta u\|_{L^2(\Omega_{R,R'})} \leq C\left(\frac{1}{R^2}\|u\|_{L^2(\Omega_{R,R'})} + \frac{1}{R}\|\nabla u\|_{L^2(\Omega_{R,R'})}\right),$$

where $C > 0$ is independent of R and R'.

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Remark

In the study of ocean flows: $R' = (1 + \varepsilon)R$.

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MISSING Korn's Inequality: $\|u\|_{H^1} \leq C\|A^{1/2}u\|_{L^2} \leq C\|Au\|_{L^2}$.

Estimate of the Trilinear term $\langle (u \cdot \nabla)u, Au \rangle$.

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In thin domain, $u = v + w$ where v is 2D-like, and w has good Poincare-like inequalities. Then we estimate

$$\begin{aligned}\langle (u \cdot \nabla) u, Au \rangle &= \langle (w \cdot \nabla) u, Au \rangle + \langle (v \cdot \nabla) u, Au \rangle \\ &= \langle (w \cdot \nabla) u, Au \rangle + \langle (v \cdot \nabla) u, Au + \Delta u \rangle \\ &\quad - \langle (v \cdot \nabla) u, \Delta u \rangle.\end{aligned}$$

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One of the above terms

$$\begin{aligned}&|\langle (v\cdot\nabla)u, Au+\Delta u\rangle| \le C\|v|\nabla u|\|_{L^2}(\varepsilon\|u\|_{H^1} + \|u\|_{L^2})\\ &\le C\{\varepsilon^{-1/4}\|u\|_{L^2}^{1/2}\|u\|_{H^1}\|u\|_{H^2}^{1/2} + \|u\|_{L^2}^{1/2}\|u\|_{H^1}^{1/2}\|u\|_{H^2}\}\\ &\quad\times(\varepsilon\|u\|_{H^1} + \|u\|_{L^2}).\end{aligned}$$

which is acceptable.

Other discussions

- Non-linear estimate
- Commutator estimate
- Other boundary conditions.

Non-linear estimate

In the Key Lemma, what do we get if Φ is $(u \cdot \nabla)u$ or related term?
Can we use this to improve the estimate of the trilinear term?

Commutator estimate

Liu-Pego proved for Dirichlet boundary condition in general Ω that

$$\|P\Delta u - \Delta P u\|_{L^2} \leq (\varepsilon + \frac{1}{2})\|u\|_{H^2} + C_\varepsilon \|u\|_{H^1}.$$

for $u \in H^2 \cap H_0^1$.

Other boundary conditions

Example: friction boundary condition on $\partial\Omega$:

$$u \cdot N = 0, \quad [(Du)N]_{\tan} + \alpha u = 0,$$

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In thin domain Ω_ε, the friction $\alpha = \alpha_\varepsilon$.
If $\alpha_\varepsilon = O(\varepsilon)$ as $\varepsilon \to 0$, the same method works.
What if $\alpha_\varepsilon \leq \alpha$? (more. . . in Spring or next Fall!)

THANK YOU FOR YOUR TIME AND ATTENTION.
(this is my first presentation with BEAMER)

Miscellaneous

Green's formula: for $u \in H^2(\Omega_\varepsilon)$ and $v \in H^1(\Omega_\varepsilon)$:

$$\int_{\Omega_\varepsilon} \Delta u \cdot v dx = \int_{\Omega_\varepsilon} -2(Du : Dv) + (\nabla \cdot u)(\nabla \cdot v) dx \\ + \int_{\partial\Omega_\varepsilon} \{2((Du)N) \cdot v - (\nabla \cdot u)(v \cdot N)\} d\sigma.$$