Lecture Notes on Fluid Dynamics

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1 Review of differential operators

Given a scalar function $\phi : \mathbb{R}^N \mapsto \mathbb{R}$, its *gradient* $\nabla\phi$ is the vector of partial derivatives

$$\nabla\phi \doteq (\phi_{x_1}\,,\ \phi_{x_2}\,,\ \ldots\ ,\phi_{x_N}) = \Big(\frac{\partial}{\partial x_1}\phi\,,\ \frac{\partial}{\partial x_2}\phi\,,\ \ldots\ ,\frac{\partial}{\partial x_N}\phi\Big)\,.$$

For a vector field $u : \mathbb{R}^N \mapsto \mathbb{R}^N$, its *divergence* is

$$\mathrm{div}\, u \doteq \sum_{i=1}^{N} u^i_{x_i}\,.$$

The *Laplace operator* is defined as

$$\Delta\phi \doteq \sum_{i=1}^{N} \frac{\partial^2\phi}{\partial x_i^2} \,=\, \mathrm{div}\,\nabla\phi\,.$$

Ňow consider a velocity field $u : \mathbb{R}\times\mathbb{R}^N \mapsto \mathbb{R}^N$, so that $u(t,x) = (u^1,\ldots,u^N)(t,x)$ denotes the velocity of the particle located at the point $x = (x_1,\ldots,x_N)$ at time t. We then define the *Material Derivative* of a function $\phi = \phi(t,x)$ as

$$D_t\phi \doteq \frac{\partial}{\partial t}\phi + \sum_{i=1}^{N} u^i \frac{\partial}{\partial x_i}\phi\,.$$

This provides the time-derivative of ϕ along particle trajectories. W̌e recall that the wedge product of two vectors $v, w \in \mathbb{R}^3$ can be obtained as follows. Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be the vectors in the standard orthonormal basis of $\mathbb{R}^3$. Then

$$v \wedge w = \det\begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}\,.$$

Given a velocity field $u : \mathbb{R}^3 \mapsto \mathbb{R}^3$, its *vorticity* $\omega = \mathrm{curl}\ u$ is defined as

$$\omega = \mathrm{curl}\, u \doteq \big(\partial_2 u^3 - \partial_3 u^2\,,\ \partial_3 u^1 - \partial_1 u^3\,,\ \partial_1 u^2 - \partial_2 u^1\big)\,.$$

Note that, formally, one can write

$$\mathrm{curl}\, u = \nabla \wedge u = \det\begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \partial_1 & \partial_2 & \partial_3 \\ u_1 & u_2 & u_3 \end{pmatrix}\,.$$

2 Euler and Navier-Stokes equations for a homogeneous incompressible fluid

The flow of a homogeneous incompressible, non-viscous fluid in $\mathbb{R}^N$ is modelled by the system of **Euler equations**

$$\begin{cases} D_t\, u(t,x) = -\nabla p(t,x), & \text{balance of momentum} \\ \operatorname{div} u(t,x) \equiv 0, & \text{incompressibility condition} \\ u(0,x) = u_0(x), & \text{initial data} \end{cases} \tag{1}$$

Where t is time,
$x \in \mathbb{R}^N$ is the Eulerian space variable,
$u = (u^1, \ldots, u^N)$ is the fluid velocity,
latex fluid.tex p is the scalar pressure,
$D_t = \partial_t + u \cdot \nabla_x$ is the material derivative.

When viscosity is present, an additional diffusion term is present in the momentum equation. This leads to the **Navier-Stokes** equations:

$$\begin{cases} D_t\, u(t,x) = -\nabla p(t,x) + \nu \Delta u, & \text{balance of momentum} \\ \operatorname{div} u(t,x) \equiv 0, & \text{incompressibility condition} \\ u(0,x) = u_0(x), & \text{initial data} \end{cases} \tag{2}$$

The positive constant ν is the *coefficient of kinematic viscosity*. Its reciprocal is called the Reynolds number: $Re = 1/\nu$.

Ǐf the fluid motion takes place in a bounded set $\Omega \subset \mathbb{R}^N$, one must add suitable boundary conditions. In the case of the inviscid Euler equations, one requires that the velocity u of the fluid is tangential to the boundary $\partial\Omega$. Calling $\mathbf{n}$ the unit outer normal, we thus have

$$\mathbf{n} \cdot u = 0 \qquad x \in \partial\Omega\,. \tag{3}$$

In the presence of viscosity, instead of (3), the Navier-Stokes equations are supplemented by the non-slip boundary conditions

$$u = 0 \qquad x \in \partial\Omega\,. \tag{4}$$

2.1 Derivation of Euler equation from the balance law of momentum

Consider any region $W_0 \subset R^n$. Denote by W_t the region occupied at time $t > 0$ by those fluid particles which are initially in W_0. Assume that the only force acting on the fluid through the boundary ∂W_t is the pressure. Then the balance law of momentum takes the form

$$\begin{aligned} \frac{d}{dt} \int_{W_t} u\, dV &= [\text{total forces across the boundary}] \\ &= -\int_{\partial W_t} p\, n\, dA. \end{aligned} \tag{5}$$

If e is any fixed vector in $\mathbb{R}^N$ by the divergence theorem we get

$$\begin{aligned} \int_{\partial W_t} p\, e \cdot n\, dA &= \int_{W_t} \operatorname{div}(p\, e)\, dV \\ &= \int_{W_t} \nabla p \cdot e\, dV. \end{aligned} \tag{6}$$

Then:

$$\frac{d}{dt}\int_{W_t} u\, dV = \int_{W_t} D_t u\, dV = -\int_{W_t} \nabla p\, dV, \tag{7}$$

for any region W_t in the fluid at time t. Hence the equality holds also in the differential form:

$$D_t\, u \;=\; -\nabla p \qquad \text{Euler equation.}$$

2.2 Symmetry groups for the Euler and Navier-Stokes equations

Assume that $u = u(t,x)$ and $p = p(t,x)$ provide a solution to Euler or Navier-Stokes equation. Further solutions can then be obtained by various variable transformations.

- *Translation Invariance*: For any constant vector c in $\mathbb{R}^N$

  $$\begin{cases} u^c(t,x) := \; u(t, x - ct) + c, \\ p^c(t,x) := \; p(t, x - ct), \end{cases}$$

  is another solution.

- *Rotation Invariance*: For any orthogonal matrix $\mathcal{Q}$

  $$\begin{cases} u^{\mathcal{Q}}(t,x) := \; \mathcal{Q}^T\, u(t, \mathcal{Q}x), \\ p^{\mathcal{Q}}(t,x) := \; p(t, \mathcal{Q}x), \end{cases}$$

  is another solution.

- *Scale Invariance*: If u, p provide a solution to the Euler equations, then for any $\lambda,\, \tau > 0$ the functions

  $$(A) \qquad \begin{cases} u^{\lambda,\,\tau}(t,x) := \; \frac{\lambda}{\tau}\, u(\frac{t}{\tau}, \frac{x}{\lambda}), \\ \\ p^{\lambda,\,\tau}(t,x) := \; \frac{\lambda^2}{\tau^2}\, p(\frac{t}{\tau}, \frac{x}{\lambda}), \end{cases}$$

  provide a 2-parameters family of solutions to the Euler equation.

  If u, p are a solution to the Navier-Stokes equations, then for all $\tau > 0$ the functions

  $$(B) \qquad \begin{cases} u^{\tau}(t,x) := \; \tau^{-1/2}\, u(\frac{t}{\tau}, \frac{x}{\tau^{1/2}}), \\ \\ p^{\tau}(t,x) := \; \tau^{-1}\, p(\frac{t}{\tau}, \frac{x}{\tau^{1/2}}). \end{cases}$$

  provide a 1-parameter family of solutions to the Navier-Stokes equation for $\tau \in \mathbb{R}$. Observe that (A) coincides with (B) when $\lambda = \tau^{1/2}$

3 Particle trajectories

Given a fluid flow with velocity field u (not necessarily incompressible), the *particle trajectory mapping*

$$\begin{array}{rcl} X \quad : [0,\infty) \times \mathbb{R}^N & \longrightarrow & \mathbb{R}^N \\ t \quad , \quad \alpha & \mapsto & X(t,\alpha) \end{array} \tag{8}$$

describes the trajectory of the particle which is initially located at point $\alpha = (\alpha_1, \ldots, \alpha_N)$ at time $t = 0$. The *Lagrangian variable* α, can be regarded as a *particle marker*. The function $X(t,\alpha)$ is determined by solving the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t} X(t,\alpha) \; = \; u(t,\, X(t,\alpha)), \\ \\ X(0,\alpha) \; = \; \alpha. \end{cases}$$

An initial domain $W_0 \subset \mathbb{R}^N$ in the fluid evolves in time to

$$X(t, W_0) = W_t = \{X(t,\alpha) |\, \alpha \in W_0\}.$$

We shall denote by $\nabla_\alpha X$ the Jacobian matrix of first order partial derivatives of X w.r.t. α, so that

$$\Big(\nabla_\alpha X(t,\alpha)\Big)_{ij} = \frac{\partial X^i}{\partial \alpha_j}(t,\alpha)\,.$$

The evolution of $\nabla_\alpha X$ along a particle trajectory is described by the linear evolution equation

$$D_t\Big(\nabla_\alpha X(t,\alpha)\Big) = \Big(\nabla_x u(t,\, X(t,\alpha)\Big) \cdot \nabla_\alpha X(t,\alpha)\,.$$

The Jacobian determinant of $X(t,\cdot)$ is denoted as

$$J(t,\alpha) = \det\,\big(\nabla_\alpha X(t,\alpha)\big).$$

Clearly $J(0,\alpha) \equiv 1$. If u is a smooth velocity field then the time evolution of J is given by

$$\frac{\partial}{\partial t} J(t,\alpha) = \Big[\operatorname{div} u(t, X(t,\alpha))\Big]\, J(t,\alpha)\,.$$

In particular, $J(t,\, X(t,\alpha)) \equiv 1$ if the flow is incompressible.

Theorem 1 ***The Trasport Formula***
Let W be an open, bounded domain in $\mathbb{R}^N$ with smooth boundary, and let X be the particle trajectory mapping of a given smooth velocity field u.
For any smooth function $f = f(t,x)$ we have

$$\frac{d}{dt}\int_{W_t} f\, dx = \int_{W_t} [f_t + \operatorname{div}(fu)]\, dx.$$

Proof

$$\begin{aligned}
\frac{d}{dt}\int_{W_t} f\, dx &= \frac{d}{dt}\int_{W} f(t, X(t,\alpha)) J(t, X(t,\alpha))\, d\alpha \\
&= \int_W \big(D_t f\, J + f\, J_t\big)\, d\alpha \\
&= \int_W \big(f_t + u\cdot\nabla f + f \operatorname{div} u\big)\, J\, d\alpha \\
&= \int_{W_t} [f_t + \operatorname{div}(fu)]\, dx.
\end{aligned}$$

□

Definition 1 *A flow X is* incompressible *if for any subregion W with smooth boundary, and for any time $t > 0$, X is volume preserving:*

$$Vol\big(X(t,W)\big) = Vol\big(W\big).$$

Applying the transport formula with $f \equiv 1$ one obtains

Proposition 1 *The following are equivalent:*

- *The fluid flow is incompressible,*
- $\operatorname{div} u = 0$,
- $J(t,\alpha) \equiv 1$

4 Vorticity

In the following, we consider a smooth velocity field $u : \mathbb{R}^3 \mapsto \mathbb{R}^3$. We define the *vorticity field* $\omega =$ as the curl of the velocity:

$$\omega = \text{curl}\, u \doteq \left(\partial_2 u^3 - \partial_3 u^2\,,\ \partial_3 u^1 - \partial_1 u^3\,,\ \partial_1 u^2 - \partial_2 u^1\right).$$

4.1 Local behavior of an incompressible flow

In a neighborhood of any point x_0, we now show that, up to higher order terms w.r.t. $|x - x_0|$, a smooth incompressible velocity field $u = u(x)$ can be written in a unique way as the sum of **infinitesimal translation, rotation** and **deformation** field.

Indeed, let x_0 be a given point in $\mathbb{R}^3$. A first order Taylor expansion of u yields

$$u(x_0 + h) = u(x_0) + \nabla u(x_0) \cdot h + \mathcal{O}(h^2).$$

The Jacobian matrix ∇u, can be written as the sum of its symmetric and antisymmetric parts:

$$\nabla u = \underbrace{\mathcal{D}}_{\text{Symm.}} + \underbrace{\Omega}_{\text{Antisymm.}}$$

Where

$$\mathcal{D} = \tfrac{1}{2}\Big(\nabla u \,+\, (\nabla u)^t\Big), \qquad \text{hence } \mathbf{tr}\,\mathcal{D} = 0$$

$$\Omega = \tfrac{1}{2}\Big(\nabla u \,-\, (\nabla u)^t\Big), \qquad \text{hence } \Omega h = \tfrac{1}{2}\omega \wedge h.$$

Observe that, since every symmetric matrix can be diagonalized and the trace is invariant under orthogonal trasformations, $\mathcal{D}$ can be written as

$$\mathcal{D} = \begin{pmatrix} \gamma_1 & 0 & 0 \\ 0 & \gamma_2 & 0 \\ 0 & 0 & -(\gamma_1 + \gamma_2) \end{pmatrix}$$

The Taylor expansion thus takes the form

$$u(x_0 + h) = \underbrace{u(x_0)}_{\text{translation}} + \underbrace{\frac{1}{2}\omega \wedge h}_{\substack{\text{rotation by} \\ \text{vector } \omega}} + \underbrace{\mathcal{D}h}_{\text{deformation}} + \mathcal{O}(h^2).$$

4.2 Evolution equation for the vorticity

Proposition 2 *Let the velocity field $u = u(t, x)$ provide a solution to the Navier-Stokes equations. Then the vorticity $\omega = curl u$ evolves according to the equation*

$$D_t \omega \;=\; (\omega \cdot \nabla) u + \nu \Delta \omega. \tag{9}$$

Proof We use the vector identity:

$$\frac{1}{2}\nabla |u|^2 \;=\; u \wedge \text{curl}\; u + (u \cdot \nabla) u$$

in order to replace the term $(u \cdot \nabla)u$ in the Navier-Stokes equation:

$$\partial_t u + \frac{1}{2}\nabla|u|^2 - u \wedge \omega \;=\; -\nabla p + \nu\Delta u.$$

We now recall that $\mathrm{curl}(\nabla\varphi) = 0$ for every φ. Taking the curl of both sides of the above identity one obtains

$$\partial_t \omega - \mathrm{curl}\,(u \wedge \omega) \;=\; \nu\Delta\omega. \tag{10}$$

In order to write esplicitly the term $\mathrm{curl}\,(u \wedge \omega)$ we recall the following general vector identity:

$$\mathrm{curl}\,(u \wedge \omega) = (\omega \cdot \nabla)u - \omega\,\mathrm{div}\,u - (u \cdot \nabla)\omega + u\,\mathrm{div}\,\omega\,.$$

Since $\mathrm{div}\,u \equiv 0$, this reduces to

$$\mathrm{curl}\,(u \wedge \omega) = (\omega \cdot \nabla)u - (u \cdot \nabla)\omega\,.$$

Therefore, from (10) we obtain

$$\partial_t \omega + (u \cdot \nabla)\omega \;=\; (\omega \cdot \nabla)u + \nu\Delta\omega. \tag{11}$$

□

5 Vorticity transport formula for the Euler equation

The map $t \mapsto X(t, \alpha)$ describes the trajectory of the particle that was in α at $t = 0$. Consider any vector $h_0 \in \mathbb{R}^3$. At a given time t, the position of the particle which is initially at $\alpha + \varepsilon h$ can be described by

$$X(t,\, \alpha + \varepsilon h) = X(t, \alpha) + \varepsilon \cdot \nabla_\alpha X(t, \alpha) \cdot h + \mathcal{O}(\varepsilon^2)\,.$$

In the present section we will show that the inviscid vorticity equation, i.e. the vorticity evolution equation associated to Euler equation,

$$D_t\omega \;=\; (\omega \cdot \nabla)u, \tag{12}$$

can be integrated by means of the particle trajectory equation

$$\frac{d}{dt}X(t, \alpha) \;=\; u(t,\, X(t, \alpha)).$$

Let $h(t, x)$ be a smooth vector field in $\mathbb{R}^N$, and consider the perturbed flow

$$X(t, \alpha + h) = X(t, \alpha) + \underbrace{\nabla_\alpha X(t, \alpha)\, h}_{(\star)} + \mathcal{O}(h^2).$$

$(\star)$ is the term which tell us how the perturbation evolves in time. Then the evolution equation of a first order perturbation takes the form

$$D_t\, h \;=\; h \cdot \nabla u.$$

The following lemma applies also to the more general case in which the velocity field u is not divergence free.

Lemma 1 *Let $u = u(t,x)$ be a smooth vector field with associed particle trajectory mapping $X(t,\alpha)$. The smooth vector field $h = h(t,x)$ satisfies*

$$D_t\, h \;=\; h\cdot\nabla u \tag{13}$$

if and only if

$$h(t, X(t,\alpha)) = \nabla_\alpha X(t,\alpha) h(0,\alpha). \tag{14}$$

Proof The particle trajectory mapping is the solution of

$$\begin{cases} \frac{d}{dt} X(t,\alpha) \;=\; u(t,\, X(t,\alpha)), \\ X(0,\alpha) \;=\; \alpha. \end{cases}$$

By differentiation with respect to α we obtain

$$\frac{d}{dt}\nabla_\alpha X(t,\alpha) \;=\; \nabla_x u(t,\, X(t,\alpha))\,\nabla_\alpha X(t,\alpha),$$

hence

$$\frac{d}{dt}\nabla_\alpha X(t,\alpha)\, h(0,\alpha) = \nabla_x u(t,\, X(t,\alpha))\,\nabla_\alpha X(t,\alpha)\, h(0,\alpha)\,,$$

If (14) holds then

$$\frac{d}{dt} h(t, X(t,\alpha)) \;=\; \nabla_x u(t,\, X(t,\alpha))\, h(t, X(t,\alpha))$$

This implies (13). Viceversa, if (13) holds, then for a fixed α the vector $h(t,\, X(t,\alpha))$ satisfies

$$\frac{d}{dt} h(t, X(t,\alpha)) \;=\; \nabla_x u(X(t,\alpha))\, h(t, X(t,\alpha)).$$

Since $t \mapsto h(t, X(t,\alpha))$ and $t \mapsto \nabla_\alpha X(t,\alpha) h(0,\alpha)$ satisfy the same linear ODE with initial data $h(0,\alpha)$, they coincide for all times t. □

Proposition 3 ***Vorticity Transport Formula***

Let $X(t,\alpha)$ be the smooth particle trajectory mapping corresponding to the smooth divergence-free vector field $u(t,x)$. Then

$$\omega\big(t,\, X(t,\alpha)\big) \;=\; \nabla_\alpha X(t,\alpha)\omega_0(\alpha)$$

solves (12).

Proof We observe that the vorticity equation has the same form of the evolution equation satisfied by a first order perturbation.

The proof of Proposition 3 follows from the application of the lemma in the case $h = \omega$. □

Definition 2 *The smooth curve*

$$\gamma = \{\gamma(s) \in \mathbb{R}^N \,|\, 0 < s < 1\}$$

is a vortex line *at time t if*

$$\frac{d}{ds}\gamma(s) = \lambda(s)\omega(t, \gamma(s)), \qquad \textit{for some } \lambda(s) \neq 0.$$

Corollary 1 *In a non-viscous fluid flow, the vortex lines move with the fluid.*

Proof Let γ_0 be a vortex line at time $t = 0$. As a set of particles γ_0 evolves in time to γ_t

$$\gamma_t := \{X(t, \gamma_0(s)) \,|\, 0 < s < 1\},$$

and there holds

$$\begin{aligned} \frac{d}{ds}\gamma_t(s) &= \nabla_\alpha X(t, \gamma_0(s)) \frac{d}{ds}\gamma_0(s) \\ &= \nabla_\alpha X(t, \gamma_0(s))\lambda(s)\omega(0, \gamma_0(s)) \\ &= \lambda(s)\omega(t, X(t, \gamma_0(s))) \\ &= \lambda(s)\omega(t, \gamma_t(s)) \end{aligned} \tag{15}$$

□

Definition 3 *Let u be a smooth divergence-free vector field and let γ be a smooth oriented closed curve in the fluid. The* **circulation** *of u around γ the line integral*

$$\oint_\gamma u \cdot dl.$$

Lemma 2 ***Transport theorem for curves***

Let $\gamma_t := X(t, \gamma_0)$ be a closed curve transported by the flow. The following identity holds:

$$\frac{d}{dt}\oint_{\gamma_t} u \cdot dl \,=\, \oint_{\gamma_t} D_t u \cdot dl.$$

Proof Let $s \mapsto \gamma_0(s)$ be a parametrization of γ_0, and let $s \mapsto X(t, \gamma_0(s)) = \gamma_t(s)$ be the parametrization of γ_t.

$$\begin{aligned} \frac{d}{dt}\oint_{\gamma_t} u \cdot dl \,&=\, \frac{d}{dt}\int_0^1 u(t, X(t, \gamma_0(s)))\frac{\partial}{\partial s}X(t, \gamma_0(s))\, ds \\ &= \int_0^1 D_t u(t, X(t, \gamma_0(s)))\frac{\partial}{\partial s}X(t, \gamma_0(s))\, ds \\ &\quad + \int_0^1 u(t, X(t, \gamma_0(s)))\frac{\partial}{\partial t}\frac{\partial}{\partial s}X(t, \gamma_0(s))\, ds \\ &= \oint_{\gamma_t} D_t u \cdot dl \,+\, \int_0^1 \frac{\partial}{\partial s}\frac{|u(t, X(t, \gamma_0(s)))|^2}{2}\, ds. \end{aligned} \tag{16}$$

Since γ_t is a loop the last term of the right hand side is null and we obtain the equality

$$\frac{d}{dt}\oint_{\gamma_t} u \cdot dl \,=\, \oint_{\gamma_t} D_t u \cdot dl.$$

□

Theorem 2 ***Kelvin's circulation theorem***

For an inviscid, incompressible fluid the circulation of the velocity u around any closed curve γ_t moving with the fluid is constant in time.

Proof

$$\frac{d}{dt}\oint_{\gamma_t} u \cdot dl \;=\; \oint_{\gamma_t} D_t u \cdot dl = -\oint_{\gamma_t} \nabla p \, dl\,. \tag{17}$$

since the line integral of a gradient on a closed loop is zero. □

Corollary 2 *The vorticity flux through a 2-dimensional surface moving with the fluid is constant in time.*

Consider the 2-dimensional surface Σ_t whose boundary is a closed curve γ_t moving with the fluid. By Stokes formula we obtain

$$\oint_{\gamma_t} u \cdot dl \;=\; \int_{\Sigma_t} \omega \cdot dA,$$

where n is a vector, $\|u\| = 1$, normal to Σ_t

An example of this behavior is given by tornadoes: the surface of a horizontal section decreases as the distance from the soil, then the angular velocity around the vertical axis has to increase.

6 Conserved quantities for the Euler equation

A conservation law in N space dimensions takes the form:

$$\partial_t u(t,x) + \mathrm{div}_x F(u(t,x)) = 0\,, \tag{18}$$

where $t \geqslant 0$ is the time variable, $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ is the space variable, $u = (u_1, \dots, u_n)$ is a vector in $\mathbb{R}^N$ and F is a smooth map defined on a convex neighborhood of the origin in $\mathbb{R}^N$ with values in $\mathbb{R}^N$. The components of u are called the *conseved quantities* while the components of F, are the *fluxes*.

Let $W \subset \mathbb{R}^N$ be a closed region with smooth boundary. Integrating (18) on W we obtain:

$$\begin{aligned} \frac{d}{dt}\int_W u(t,x)\,dx &= \int_W u_t(t,x)\,dx \\ &= -\int_W \mathrm{div}\; F\big(u(t,x)\big)\,dx \\ &= -\int_{\partial W} F\big(u(t,x)\big)\cdot n\,dA \\ &= \text{[Flow through the boudary]}. \end{aligned}$$

Observation 1 *Notice that*

$$\frac{d}{dt}\int_{\mathbb{R}^N} u(t,x)\,dx \;=\; 0 \qquad \textit{if} \quad |F| \;=\; \wr(|x|^{1-N}).$$

In fact

$$\frac{d}{dt}\int_{\mathbb{R}^N} u(t,x)\,dx = \lim_{R\to\infty}\frac{d}{dt}\int_{|x|\leqslant R} u(t,x)\,dx = \lim_{R\to\infty} -\int_{|x|=R} F(u(t,x))\cdot n\,dA,$$

is zero if $|F| \;=\; o(|x|^{1-N})$.

The following are conserved quantities for the Euler equations.

- **Components of momentum** The Euler equation is equivalent to the momentum balance law. Then, if u is a solution to the Euler equation its components satisfy

$$u^i_t + \sum_{j=1}^N u^j u^i_{x_j} \; = \; -p_{x_i},$$

which can be written (div u=0) as a system of conservation law:

$$u^i_t + \text{div }(u^i\, u \;+\; P_i) = 0, \qquad\qquad i = 1, \dots, N$$

where $P_i \equiv (0, \dots, 0, p_i, 0, \dots, 0)$.

- **Vorticity** The i-th component of the vorticity equation is

$$\omega^i_t \;+\; (u\nabla)\omega^i \;=\; (\omega\nabla)u^i.$$

It can be written as a system of conservation laws::

$$\omega^i_t \;+\; \text{div }(\omega^i\, u \;+\; u^i\, \omega) = 0.$$

- **The energy** By multipling Euler equation by u we obtain the *energy equation*

$$\Big(\frac{|u|^2}{2}\Big)_t \;+\; \sum_{i,j=1}^N u^i u^j u^i_{x_j} \;+\; \sum_{i=1}^N u^i p_{x_i} = 0.$$

It can be written as a conservation law

$$\Big(\frac{|u|^2}{2}\Big)_t \;+\; \text{div }\Big(\frac{|u|^2}{2}\, u \;+\; p\, v\Big) = 0.$$

- **Helicity** The helicity measures the component of velocity in the direction of the vorticity:

$$H \doteq u \cdot \omega.$$

$$\begin{aligned}
\frac{\partial}{\partial t} H \;&=\; u_t \cdot \omega \;+\; u \cdot \omega_t \\
&=\; [-(u\cdot\nabla)u \;-\; \nabla p\,]\cdot\omega \;+\; u\cdot[-(u\cdot\nabla)\omega \;+\; (\omega\cdot\nabla)u\,] \\
&=\; -\sum_{i,j=1}^N u^j u^i_{x_j}\omega^i \;-\; \sum_{i=1}^N p_{x_i}\omega^i \;-\; \sum_{i,j=1}^N u^i u^j \omega^i_{x_j} \;+\; \sum_{i,j=1}^N u^i \omega^j u^i_{x_j}
\end{aligned}$$

This can be written as

$$H_t \;+\; \text{div }\Big[(\omega\cdot u)u \;-\; \frac{|u|^2}{2}\omega \;+\; p\,\omega\Big] \;=\; 0.$$

7 Leray formulation of Navier-Stokes equation

Our aim in this section will be to eliminate the pressure from the explicit formulation of Navier-Stokes equation in order to obtain a system of closed evolution equations for u. Let's assume that u satisfies the equation

$$\frac{du}{dt} = -(u \cdot \nabla)u + \nu \Delta u - \nabla p \tag{19}$$

Then u solves (2) if and only if the function p has been chosen in such a way that $\operatorname{div} u(t,x) = 0$ for all $t \geqslant 0$.

We define a subspace $E \subset \mathbb{L}^2(\mathbb{R}^3;\mathbb{R}^3)$ containing all vector fields whose divergence (in distributional sense) vanishes identically.

$$E \doteq \{u \in \mathbb{L}^2; \ \ \operatorname{div}\, u = 0\}.$$

Assume that $t \mapsto u(t) \in E$ is a smooth solution of (19) Observe that

$$\begin{aligned} \operatorname{div} u = 0 \ &\Rightarrow \ \operatorname{div}(\Delta u) = 0, \\ \operatorname{div} u = 0 \ &\not\Rightarrow \ \operatorname{div}(u \cdot \nabla u) = 0. \end{aligned} \tag{20}$$

More precisely, if the i-th component of $(u \cdot \nabla)u$ is

$$[(u \cdot \nabla)u]^i = \sum_j u^j u^i_{x_j},$$

we can write

$$\operatorname{div}\ (u \cdot \nabla)u = \sum_{i,j} \big(u^j_{x_i} u^i_{x_j} + \underbrace{u^j u^i_{x_i x_j}}_{A_{i,j}} \big).$$

Since $\operatorname{div} u = \sum_i u^i_{x_i} = 0$, we get $\sum_{i,j} A_{i,j} = 0$. Therefore, in general,

$$\operatorname{div}(u \cdot \nabla)u = \sum_{i,j} u^j_{x_i} u^i_{x_j} = \operatorname{tr}(\nabla u)^2 \neq 0.$$

We now need to choose p so that the sum

$$\frac{du}{dt} = \underbrace{-(u \cdot \nabla)u}_{\notin E} + \underbrace{\nu \Delta u}_{\in E} - \nabla p$$

lies in E.

Since

$$\begin{aligned} \operatorname{div}\big[(u \cdot \nabla)u + \nabla p\big] &= \sum_{i,j} u^j_{x_i} u^i_{x_j} + \Delta p \\ &= \operatorname{tr}(\nabla u)^2 + \Delta p, \end{aligned}$$

the pressure p satisfies the relation

$$\operatorname{div}\big[(u \cdot \nabla)u + \nabla p\big] = 0$$

if and only if it solves the elliptic equation:

$$\Delta p = -\operatorname{tr}(\nabla u)^2 \tag{21}$$

on the whole space $\mathbb{R}^N$.

We now recall a solution formula for linear elliptic equations

Lemma 3 ***Solution of the Poisson equation***
Let f be a smooth function in $\mathbb{R}^N$, vanishing sufficiently rapidly as $|x| \to \infty$. Then the solution to the Poisson equation

$$\begin{cases} \Delta z \,=\, f, \\ \nabla z \to 0, \quad \text{as } |x| \to 0, \end{cases}$$

is given by

$$z(x) \,=\, \mathcal{N}_N * f(x) \,=\, \int_{\mathbb{R}^N} \mathcal{N}_N(x-y) f(y)\, dy,$$

where the fundamental solution $\mathcal{N}$ is the Newton potential

$$\mathcal{N}_N(x) = \begin{cases} \frac{1}{2\pi} \ln |x|, & \text{if } N = 2, \\ \frac{1}{(2-N)\,\sigma_N} |x|^{2-N}, & \text{if } N \geqslant 3 \end{cases} \tag{22}$$

and σ_N denotes the surface area of a unit sphere in $\mathbb{R}^N$.

Observation 2 *$\mathcal{N}_N$ is the solution in $\mathbb{R}^N$ of*

$$\Delta \mathcal{N} = \delta_0$$

where δ_0 is the Dirac mass concentrated at the origin.

By applying this Lemma we can solve the equation (21)

$$p(t,x) \,=\, \int_{\mathbb{R}^N} \mathcal{N}_N(x-y)\, \mathrm{tr}(\nabla u(t,y))^2\, dy.$$

The gradient of p takes the form:

$$\nabla p(t,x) \,=\, \nabla \mathcal{N}_N * \mathrm{tr}(\nabla u)^2\,(x) \,=\, C_N \int_{\mathbb{R}^N} \frac{x-y}{|x-y|^N}\, \mathrm{tr}(\nabla u(t,y))^2\, dy,$$

for a suitable constant C_N. Using this result we can eliminate the pressure from the explicit formulation of Navier-Stokes equation and we obtain a system of closed evolution equations for u. This is the Leray's formulation of Navier-Stokes equation

$$D_t u(t,x) \,=\, \nu \Delta u \,-\, C_N \int_{\mathbb{R}^N} \frac{x-y}{|x-y|^N}\, \mathrm{tr}(\nabla u(t,y))^2\, dy. \tag{23}$$

Observation 3 *The Navier-Stokes equation can be written:*

$$u_t \,=\, \Pi_E (u \cdot \nabla) u \,+\, \nu \Delta u,$$

where Π_E is the perpendicular projection operator on the space E.

To prove this claim, we have to show that, for any scalar function p vanishing sufficiently fast as $|x| \to \infty$, the gradient ∇p is orthogonal to all functions $v \in E$.

Lemma 4 *Let w be a smooth, divergence-free vector field in $\mathbb{R}^N$ and let q be a smooth scalar function such that*

$$|w(x)||q(x)| = o(|x|^{1-N}) \qquad \text{as } |x| \to \infty.$$

Then w and ∇q are orthogonal in $\mathbb{L}^2$:

$$\int_{\mathbb{R}^N} w \cdot \nabla q \, dx \,=\, 0.$$

Proof A direct computation yields

$$\int_{|x|\leq R} \text{div } (w\, q)\, dx \;=\; \lim_{R\to\infty} \int_{|x|=R} (w\, q) \cdot n\, dA \;=\; 0$$

□

8 Vorticity-stream formulation of the Navier-Stokes equations

Taking the curl of both sides of Navies-Stokes equation we obtain the vorticity evolution equation, in which the pressure does not appear. This provides an alternative way to close our system. In this case, the key step is to recover u from $\omega \doteq \text{curl } u$).

8.1 The 2-dimensional case

If the flow lies on a plane, the velocity field has components $u = (u^1,\, u^2,\, 0)$ and the vorticity field is $\omega = (0,\, 0,\, u^2_{x_1} - u^1_{x_2})$. We can denote them respectively as

$$u = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} \qquad \text{and} \qquad \omega = u^2_{x_1} - u^1_{x_2}.$$

In this case the evolution of the vorticity is reduced to a scalar equation

$$\partial_t \omega \;+\; (u\cdot\nabla)\omega \;=\; \nu\Delta\omega.$$

To recover the velocity u from the scalar function ω, we first recall that the system of PDE's

$$\begin{cases} \psi_{x_1} \;=\; a(x_1, x_2), \\ \psi_{x_2} \;=\; b(x_1, x_2), \end{cases}$$

has a solution if and only if $a_{x_2} \;=\; b_{x_1}$. Since $\text{div } u = u^1_{x_1} + u^2_{x_2} = 0$, there exist a function ψ, called **stream function** , which solves

$$\begin{cases} \psi_{x_1} \;=\; u^2(x_1, x_2), \\ \psi_{x_2} \;=\; -u^1(x_1, x_2). \end{cases}$$

We then have

$$u = \begin{pmatrix} -\psi_{x_2} \\ \psi_{x_1} \end{pmatrix} \doteq \nabla^\perp \psi, \tag{24}$$

$$\omega = \text{curl}\, u = \text{curl}\,(\nabla^\perp\psi) = \Delta\psi. \tag{25}$$

Let $\omega : \mathbb{R}^2 \mapsto \mathbb{R}$ be a given vorticity function. We can determine ψ by solving the Poisson equation (25) as in lemma 3

$$\psi(t,x) \;=\; \mathcal{N}_2 * \omega(x) \;=\; \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln(|x-y|)\omega(t,y)\, dy\,.$$

We then obtain u from (24),

$$u(t,x) \;=\; \mathcal{K}_2 * \omega(x) \;=\; \int_{\mathbb{R}^2} \mathcal{K}_2(x-y)\omega(y)\, dy. \tag{26}$$

Here the kernel $\mathcal{K}_2$ is the fundamental solution of the equation

$$\operatorname{curl} z \;=\; \delta_0,$$

its explicit form is

$$\mathcal{K}_2(x) = \frac{1}{2\pi\,|x|^2}\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix},$$

Notice that the singularity at the origin has size

$$|\mathcal{K}_2(x)| = \frac{1}{2\pi\,|x|}\,.$$

The formula (26) is called *Biot-Savart equation* since it takes the same form of the Biot-Savart law for the magnetic field. Observe that also in this second reformulation of the Navier-Stokes Equation the pressure can be obtained from the the Poisson equation

$$\Delta p = -\mathrm{tr}(\nabla u)^2.$$

8.2 The 3-dimensional case

Since in the 3-dimensional case the term $\omega \cdot \nabla u$ in the vorticity equation is non zero, we have to recover both u and ∇u from vorticity.

Lemma 5 *Let $\omega : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field with a sufficiently fast decay as $|x| \to \infty$. The overdetermined elliptic system*

$$\begin{cases} \operatorname{curl} u \;=\; \omega, \\ \operatorname{div} u \;=\; 0, \end{cases} \tag{27}$$

admits a solution if and only if $\operatorname{div}\omega = 0$.

In the positive case, the solution is constructed by means of the equation

$$u \;=\; -\operatorname{curl}\psi. \tag{28}$$

where the **stream function** ψ *solves the Poisson equation*

$$\Delta\psi \;=\; \omega\,. \tag{29}$$

Observation 4 *We solve the equation (29) using lemma 3*

$$\psi(x) \;=\; \mathcal{N}_3 * \omega(x) \;=\; \frac{1}{2\pi}\int_{\mathbb{R}^3} \frac{1}{|x-y|}\omega(y)\,dy,$$

and then the explicit solution to (28) is provided by the Biot-Savart formula

$$u(x) \;=\; \mathcal{K}_3 * \omega \;=\; \int_{\mathbb{R}^3} \mathcal{K}_3(x-y)\omega(y)\,dy,$$

where the kernel $\mathcal{K}_3$ is defined by the relation

$$\mathcal{K}_3(x)\cdot h \;=\; \frac{1}{4\pi}\,\frac{x \wedge h}{|x|^3} \qquad\qquad h \in \mathbb{R}^3.$$

Proof

Necessity: If $\omega = \operatorname{curl} u$ then $\operatorname{div} \omega = \operatorname{div}(\operatorname{curl} u) = 0$.

Sufficiency: Assume that $\operatorname{div} \omega = 0$. Consider the vector identity

$$\Delta\psi = \nabla(\operatorname{div}\psi) - \operatorname{curl}(\operatorname{curl}\psi). \tag{30}$$

Take the inner product with $\nabla(\operatorname{div}\psi)$ we obtain

$$\int \nabla(\operatorname{div}\psi)\cdot\Delta\psi\,dx = \int \nabla(\operatorname{div}\psi)\cdot\nabla(\operatorname{div}\psi)\,dx - \int \nabla(\operatorname{div}\psi)\cdot\operatorname{curl}(\operatorname{curl}\psi)\,dx.$$

The first and the third integral terms vanish because divergence-free fields are orthogonal to gradient fields, as shown in lemma 4. Therefore

$$\int_{\mathbb{R}^N} |\nabla(\operatorname{div}\psi)|^2\,dx = 0.$$

This proves that ψ is divergence-free and

$$\omega = \Delta\psi = -\operatorname{curl}(\operatorname{curl}\psi).$$

Setting

$$u \doteq -\operatorname{curl}\psi$$

we obtain $\omega = \operatorname{curl} u$ and of course $\operatorname{div} u = \operatorname{div}(-\operatorname{curl}\psi) = 0$. □

9 Particle trajectory formulation of the Euler equation

In this section we show that the Euler equations can be reformulated as a family of integro-differential equation for the particle trajectories. Given a smooth velocity field u, the particle trajectory mapping X satisfies the following Cauchy problem, for each value of the Lagrangian variable $\alpha \in \mathbb{R}^N$

$$\begin{cases} \frac{d}{dt}X(t,\alpha) = u(t, X(t,\alpha)), \\ X(0,\alpha) = \alpha. \end{cases}$$

We would like to recover u from X and ∇X. We begin by recalling *the vorticity transport formula*

$$\omega(t, X(t,\alpha)) = \nabla_\alpha X(t,\alpha)\omega_0(\alpha),$$

and the *Biot-Savart law*

$$u(t,x) = \int_{\mathbb{R}^N} \mathcal{K}_N(x-y)\omega(t,y)\,dy.$$

Together, these equations yield

$$u(t, X(t,\alpha)) = \frac{d}{dt}X(t,\alpha) = \int_{\mathbb{R}^N} \mathcal{K}_N(X(t,\alpha)-y)\omega(t,y)\,dy.$$

Consider the change of integration variable given by

$$y = X(t,\alpha'), \quad \text{for a suitable Lagrangian variable } \alpha',$$

Then

$$u(t,\, X(t,\alpha)) \;=\; \int_{\mathbb{R}^N} \mathcal{K}_N(X(t,\alpha) - X(t,\alpha'))\omega(t, X(t,\alpha'))\, d\alpha'$$
$$\;=\; \int_{\mathbb{R}^N} \mathcal{K}_N(X(t,\alpha) - X(t,\alpha'))\nabla_\alpha X(t,\alpha')\omega_0(\alpha')\, d\alpha'.$$

The explicit form of this integral depends on N, we will consider the two cases separately.

The 3-dimensional case

In the 3-dimensional case the kernel $\mathcal{K}_3$ is defined by the relation

$$\mathcal{K}_3(x) \cdot h \;=\; \frac{1}{4\pi}\,\frac{x \wedge h}{|x|^3} \qquad\qquad \text{for } x,\;\, h \in \mathbb{R}^3.$$

Then the integral form of $u(t, X(t,\alpha))$

$$u(t, X(t,\alpha)) \;=\; \int_{\mathbb{R}^3} \mathcal{K}_3(X(t,\alpha) - X(t,\alpha'))\nabla_\alpha X(t,\alpha')\omega_0(\alpha')\, d\alpha'$$

is a singular integral operator in a neighborhood of the origin.

The 2-dimensional case

In this case the situation is simpler because vorticity is constant along particle trajectories

$$\omega(t, X(t,\alpha)) \;=\; \omega_0(\alpha).$$

In fact if the flow lies on the plane $\mathbb{R}^2$, then the velocity field is $u = (u^1, u^2, 0)^t$, the vorticity is $\omega = (0, 0, u^2_{x_1} - u^1_{x_2})^t$, and we get:

$$\nabla_\alpha X(t,\alpha)\,\omega = \begin{pmatrix} \frac{dX_1}{d\alpha_1} & \frac{dX_1}{d\alpha_2} & 0 \\ \frac{dX_2}{d\alpha_1} & \frac{dX_2}{d\alpha_2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \omega_3 \end{pmatrix}.$$

Then the integral form of $u(t, X(t,\alpha))$ becomes:

$$u(t, X(t,\alpha)) \;=\; \int_{\mathbb{R}^2} \mathcal{K}_2(X(t,\alpha) - X(t,\alpha'))\omega_0(\alpha')\, d\alpha'.$$

10 Energy estimates

Taking the scalar product of the Navier-Stokes equation with u and integrating by parts we obtain:

$$\frac{d}{dt}\int_{\mathbb{R}^N} \frac{|u|^2}{2}\, dx \;+\; \int_{\mathbb{R}^N} \mathrm{div}\left(\frac{|u|^2}{2}\, u\right) dx \;=\; -\int_{\mathbb{R}^N} \mathrm{div}\,(p\, u)\, dx \;-\; \nu \int_{\mathbb{R}^N} |\nabla u|^2\, dx.$$

If we assume, as in the previous sections, that u decays sufficiently fast as $|x| \to \infty$, then the two integrals containing the divergence operator are zero because of Stokes theorem. We obtain an equality between the *time derivative of the kinetic energy* of the fluid and a *negative term depending on the viscosity coefficient*

$$\frac{d}{dt}\int_{\mathbb{R}^N} \frac{|u|^2}{2}\, dx \;=\; -\,\nu \int_{\mathbb{R}^N} |\nabla u|^2\, dx \;\leqslant\; 0.$$

This equation provides an *a priori* bound of the $\mathbb{L}^2$ norm of the velocity.

11 Uniqueness of solutions

Let v_1 and v_2 be two smooth solutions of the same Navier-Stokes equation, vanishing sufficiently fast as $|x| \to \infty$, and so, in particular, both in $\mathbb{L}^2$. If p_1 and p_2 are the associate pressures we define

$$\tilde{v} \doteq v_1 - v_2,$$
$$\tilde{p} \doteq p_1 - p_2.$$

These two functions $\tilde{v}$, $\tilde{p}$ satisfy the *evolution equation*

$$\tilde{v}_t + (v_1 \cdot \nabla)\,\tilde{v} + (\tilde{v} \cdot \nabla)\,v_2 = -\nabla\tilde{p} = \nu\Delta\tilde{v}.$$

Let's note that the linear terms involve only $\tilde{v}$ and $\tilde{p}$.

Taking the inner product of this equations with $\tilde{v}$ we obtain:

$$\begin{aligned}\int_{\mathbb{R}^N} \tilde{v} \cdot \tilde{v}_t\, dx + \underbrace{\int_{\mathbb{R}^N} \tilde{v} \cdot (v_1 \cdot \nabla)\tilde{v}\, dx}_{(\star)} + \int_{\mathbb{R}^N} \tilde{v} \cdot (\tilde{v} \cdot \nabla) v_2\, dx = \\ - \underbrace{\int_{\mathbb{R}^N} \tilde{v} \cdot \nabla\tilde{p}\, dx}_{(\star\star)} - \nu \int_{\mathbb{R}^N} \tilde{v} \cdot \Delta\tilde{v}\, dx\end{aligned} \tag{31}$$

We observe that $(\star)$ and $(\star\star)$ are equal to zero: $(\star\star)$ is the scalar product in $\mathbb{L}^2$ of a divergence free vector field and the gradient of a scalar function, while integrating $(\star)$ by parts we get

$$(\star) = \sum_{i=1}^{N} \int_{\mathbb{R}^N} v_1 \cdot \nabla\Big(\frac{|\tilde{v}^i|^2}{2}\Big)\, dx = \sum_{i=1}^{N} \int_{\mathbb{R}^N} \operatorname{div}\Big(v_1 \cdot \frac{|\tilde{v}^i|^2}{2}\Big)\, dx = 0$$

because of the fast decay of v_1 as $|x| \to \infty$.

Integrating by parts the other terms the equation (31) becomes:

$$\frac{d}{dt} \int_{\mathbb{R}^N} \frac{|\tilde{v}|^2}{2}\, dx = - \int_{\mathbb{R}^N} \tilde{v} \cdot (\tilde{v} \cdot \nabla) v_2\, dx \underbrace{- \nu \int_{\mathbb{R}^N} |\nabla\tilde{v}|^2\, dx}_{\leqslant 0}.$$

Then the following estimate holds:

$$\frac{d}{dt} \|\tilde{v}\|^2_{\mathbb{L}^2} \leqslant \|\tilde{v}\|^2_{\mathbb{L}^2} \|\nabla v_2\|_{L^\infty}.$$

Comparison with an ODE

The function defined as

$$z(T) = \exp\Big(\int_0^T 2\|\nabla v_2(t)\|_{\mathbb{L}^2}\, dt\Big) z(0)$$

solves the ODE with initial data:

$$\begin{cases} \dot{z}(t) = 2\|\nabla v_2\|_{L^\infty} z(t), \\ z(0) = \|\tilde{v}(0)\|^2_{\mathbb{L}^2}, \end{cases}$$

and for any instant $T \geqslant 0$ there holds:

$$\|\tilde{v}(T)\|^2_{\mathbb{L}^2} \leqslant z(T).$$

So if the solutions v_1 and v_2 coicide at $t = 0$, then they coincide for all $t \geqslant 0$.