Nonhomogeneous Linear Equation with Constant Coefficients

Our goal is to find the general solution of $\boxed{Ly = (a_n D^n + a_{n-1}D^{n-1} + \cdots + a_0)y = f(x) \ (*)}$

The general solution is obtained as $\boxed{y = y_c + y_p}$ where

1. y_c is the general solution of the homogeneous (or complementary) problem, i.e. $y_h = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ where $y_1, \cdots, y_n$ are n linearly independent solutions of
$$Ly = (a_n D^n + a_{n-1}D^{n-1} + \cdots + a_0)y = 0$$
with the characteristic polynomial
$$\boxed{(a_n m^n + a_{n-1}m^{n-1} + \cdots + a_0) = 0 \quad (\dagger)}.$$
2. y_p is (any) particular solution of the nonhomogeneous problem $(*)$.

The main problem then is to find y_p.

Method of Undetermined Coefficients

The method of undetermined coefficients is only applicable if the right hand side is a sum of terms of the following form

$$p(x), \quad p(x)e^{ax}, \quad p(x)e^{\alpha x}\cos(\beta x), \quad p(x)e^{\alpha x}\sin(\beta x) \tag{1}$$

where we denote by $p(x) = Cx^m + \cdots$ a general polynomial of degree m. For a sum of such terms, y_p will also be a sum of terms each of which is computed using the following

$$(a_n D^n + a_{n-1}D^{n-1} + \cdots + a_0)y = p(x)e^{r_0 x} \quad \Rightarrow \quad y_p = x^s(A_m x^m + \cdots + A_1 x + A_0)e^{r_0 x}$$

1. $s = 0$ if r_0 is not a root of $(\dagger)$
2. $s = k$ if r_0 is a root k times of $(\dagger)$.

N.B. The above case includes the case $r_0 = 0$ in which case the right side is $p(x)$.

$$(a_n D^n + a_{n-1}D^{n-1} + \cdots + a_0)y = \begin{cases} p(x)e^{\alpha x}\cos(\beta x) \\ \text{or} \\ p(x)e^{\alpha x}\sin(\beta x) \end{cases}$$

$$\Rightarrow \ y_p = x^s(A_m x^m + \cdots + A_1 x + A_0)e^{\alpha x}\cos(\beta x) + x^s(B_m x^m + \cdots + B_1 x + B_0)e^{\alpha x}\sin(\beta x)$$

1. $s = 0$ if $r_0 = \alpha \pm i\beta$ is not a root of $(\dagger)$.
2. $s = k$ if r_0 is a root k times of $(\dagger)$