MATH2450. SECTION 010. FALL 2012.

MIDTERM EXAMINATION 2-A

Name (in CAPITALS): .................................................................

Signature: ................................................. Date: ......................

READ AND FOLLOW THESE INSTRUCTIONS
This booklet contains 7 pages, including this cover page. Check to see if any are missing.
PRINT all the requested information above, and sign your name. Put your initials on
the top of every page, in case the pages become separated.
This is a closed-book examination. No books, no notes, no formula sheets, no calculators.
Do your work in the blank spaces and back of pages of this booklet.

INSTRUCTIONS FOR MULTIPLE CHOICE PROBLEMS (Questions 1 – 4):
There are 4 multiple choice problems, worth 15 points each.
You must indicate your answers clearly.
When you have decided on a correct answer to a given question, circle the answer. Each
question has a correct answer. If you give two different answers, the question will be
marked wrong. There is no penalty for guessing.

INSTRUCTIONS FOR THE WORK OUT PROBLEMS (Questions 5 – 6):
There are 2 work-out problems, worth 20 points each.
SHOW ALL WORK. Unsupported answers will receive little credit.

AFTER YOU FINISH BOTH PARTS OF THE EXAM; turn in the whole
booklet.
PART I. Multiple Choice Problems. (Questions 1 – 4)

1. (15 points) Find the limit

\[ \lim_{(x,y) \to (0,0)} \frac{2x^2y}{x^4 + 3y^4}. \]

(a) 1/2
(b) 0
(c) 2/3
(d) \infty
(e) Does not exist

\[ y = ax \]
\[ \frac{2x^3ax}{x^4 + 3a^4x^4} = \frac{2a}{1+3a^4} \]
2. (15 points) Let

\[ f(x, y) = 2x^2y \] and \[ u = \frac{1}{\sqrt{5}} \langle -2, -1 \rangle. \]

Find the directional derivative \( D_u f(1, 1) \).

(a) \( \langle 4, 2 \rangle \)

(b) \(-2\sqrt{5}\)

(c) \(-10\)

(d) \(-2/\sqrt{5}\)

(e) \(-3/\sqrt{5}\)

\[ f_x = 4xy \]

\[ f_x(1, 1) = 4 \]

\[ f_y = 2x^2 \]

\[ f_y(1, 1) = 2 \]

\[ D_u f(1, 1) = \nabla f(1, 1) \cdot u \]

\[ = \frac{4(-2) + 2(-1)}{\sqrt{5}} \]

\[ = \frac{-10}{\sqrt{5}} = -2\sqrt{5} \]
3. (15 points) Let \( z = z(x, y) \) be a function of \( x \) and \( y \) and be defined implicitly by the equation

\[
x^2 + y^3 + xyz^2 = 1.
\]

Find the partial derivative \( \frac{\partial z}{\partial x} \).

(a) \( -\frac{2x + yz^2}{2xyz} \)

(b) \( \frac{1 - x^2 - y^3}{2xyz} \)

(c) \( -\frac{1}{yz} \)

(d) \( \frac{1 - 2x - yz^2}{2xyz} \)

(e) \( -\frac{3y^2 + xz^2}{2xyz} \)

Take \( \frac{\partial}{\partial x} \) of the equation

\[
2x + 0 + yz^2 + xy \cdot 2z \quad z_x = 0
\]

\[
z_x = \frac{-2x - yz^2}{2xyz}
\]
4. (15 points) Let \( z = \ln(3x - y^2) \), where \( x = u^2v \) and \( y = 2u - v^2 \). Use the chain rule to find \( \frac{\partial z}{\partial u} \).

(a) \( \frac{6uv}{3x - y^2} \)
(b) \( \frac{2uv + 2}{3x - y^2} \)
(c) \( \frac{2 - 2y}{3x - y^2} \)
(d) \( \frac{6uv - 4y}{3x - y^2} \)
(e) \( \frac{2uv}{3x - y^2} \)

\[
\frac{\partial^2 z}{\partial x^2} = \frac{3}{3x - y^2} \quad \frac{\partial^2 z}{\partial y^2} = \frac{-2y}{3x - y^2} \\
\frac{\partial z}{\partial u} = 2uv \quad \frac{\partial y}{\partial u} = 2
\]

\[
\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u}
\]

\[
= \frac{3 (2uv)}{3x - y^2} - \frac{2y}{3x - y^2} (2)
\]

\[
= \frac{6uv - 4y}{3x - y^2}
\]
PART II. Work Out Problems. (Questions 5 - 6)

5. (20 points) Let

\[ f(x, y) = x^2 - 4xy + \frac{4}{3}y^3 - 1. \]

Find all critical points and use the second partials test to classify each point as a relative maximum, a relative minimum or a saddle point.

\[ f_x = 2x - 4y \quad f_y = -4x + 4y^2 \]
\[ f_x = 0 \Leftrightarrow x = 2y \quad f_y = 0 \quad -x + y^2 = 0 \]
\[ \rightarrow \quad -2y + y^2 = 0 \]

Critical points:
\[ \rightarrow (0, 0) \quad \text{and} \quad (4, 2) \]
\[ x = 0, 4 \quad y = 0, 2 \]

\[ f_{xx} = 2 \quad f_{yy} = 8y \]
\[ f_{xy} = -4 \]

\[ (0, 0) \quad \rightarrow \quad D = \begin{vmatrix} 2 & -4 \\ -4 & 0 \end{vmatrix} = -16 < 0 \]

Saddle point

\[ (4, 2) \quad \rightarrow \quad D = \begin{vmatrix} 2 & -4 \\ -4 & 16 \end{vmatrix} = 16 > 0 \]
\[ f_{xx} = 2 > 0 \]
\[ \Rightarrow \quad \text{relative min} \]
6. (20 points) Let \( f(x, y) = x^2 - y^2 \) and the domain \( D \) is the triangle ABC, where the vertices are \( A(-2, -1), B(1, -1) \) and \( C(1, 2) \), that is, the domain \( D \) is the region bounded by \( y = -1, x = 1, \) and \( y = x + 1 \).

Find the absolute maximum and absolute minimum of \( f(x, y) \) over the domain \( D \).

\[
f_x = 2x, \quad f_y = -2y
\]

Critical points: \((0, 0)\)

\[
\begin{align*}
\text{On } (AB) & \quad y = -1, \\
\frac{\partial f}{\partial x} &= x^2 - 1 = F_1(x), \quad -2 \leq x \leq 1 \\
F_1'(x) &= 2x, \quad F_1'(x) = 0 \Leftrightarrow x = 0 \\
\Rightarrow \text{ point } (0, -1) \\
\end{align*}

End points: \( A, B \)

\[
\begin{align*}
\text{On } (BC) & \quad x = 1 \\
y = 1 - y^2 = F_2(y), \quad -2 \leq y \leq 2 \\
F_2'(y) &= -2y, \quad F_2'(y) = 0 \Leftrightarrow y = 0 \\
\Rightarrow \text{ point } (1, 0) \\
\end{align*}

End points: \( B, C \)

\[
\begin{align*}
\text{On } (AC) & \quad y = x + 1 \\
\frac{\partial f}{\partial x} &= x^2 - (x+1)^2 \\
&= -2x - 1 = F_3(x) \quad -2 \leq x \leq 1 \\
F_3'(x) &= -2 \neq 0 \\
\end{align*}

End points: \( A, C \)

Hence

Absolute max = 3 (at \( A \))

Absolute min = -3 (at \( C \))

Table

<table>
<thead>
<tr>
<th>((x, y))</th>
<th>(f(x, y))</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 0))</td>
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<tr>
<td>((0, -1))</td>
<td>-1</td>
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MIDTERM EXAMINATION 2-B

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(a) 0
(b) 3/5
(c) \infty
(d) 3/4
(e) Does not exist

\[ y = ax \]

\[ \frac{3x^2a^2x^2}{4x^4 + a^4x^4} = \frac{3a^2}{4 + a^4} \]
2. (15 points) Let
\[ f(x, y) = 2xy^2 \] and \( u = \frac{1}{\sqrt{5}}(-1, 2) \).
\[ |u| = 1 \]
Find the directional derivative \( D_uf(1, 1) \).

(a) 10
(b) \( (2, 6) \)
(c) \( 2\sqrt{5} \)
(d) \( -2/\sqrt{5} \)
(e) \( 4/\sqrt{5} \)

\[ f_x = 2y^3 \quad f_x(1, 1) = 2 \]
\[ f_y = 6xy^2 \quad f_y(1, 1) = 6 \]

\[ D_uf(1, 1) = \nabla f(1, 1) \cdot u \\
= 2(-1) + 6(2) \]
\[ = \frac{2(-1) + 6(2)}{\sqrt{5}} \]
\[ = \frac{10}{\sqrt{5}} = 2\sqrt{5} \]
3. (15 points) Let $z = z(x, y)$ be a function of $x$ and $y$ and be defined implicitly by the equation

$$x^2 + y^3 + xyz^2 = 2.$$ 

Find the partial derivative $\frac{\partial z}{\partial y}$.

(a) $-\frac{2x + yz^2}{2xyz}$  
(b) $\frac{2 - x^2 - y^3}{2xyz}$  
(c) $\frac{2 - 3y^2 - xz^2}{2xyz}$  
(d) $-\frac{3y^2 + xz^2}{2xyz}$  
(e) $-\frac{3y}{2xz}$

Take $\frac{\partial}{\partial y}$ of the equation

$$0 + 3y^2 + xz^2 + \frac{xy}{2}z \frac{\partial z}{\partial y} = 0$$

$$z_y = -\frac{3y^2 - xz^2}{2xyz}$$
4. (15 points) Let \( z = \ln(y^2 - 4x) \), where \( x = u^2v \) and \( y = v^2 - 3u \). Use the chain rule to find \( \frac{\partial z}{\partial v} \).

(a) \( \frac{-4u^2}{y^2 - 4x} \)

(b) \( \frac{-4u^2 + 4yv}{y^2 - 4x} \)

(c) \( \frac{u^2 + 2v}{y^2 - 4x} \)

(d) \( \frac{u^2}{y^2 - 4x} \)

(e) \( \frac{2y - 4}{y^2 - 4x} \)

\[ \frac{\partial x}{\partial v} = u^2 \quad \frac{\partial y}{\partial v} = 2v \]

\[
\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} \\
= \frac{-4}{y^2 - 4x} (u^2) + \left( \frac{2y}{y^2 - 4x} \right) \cdot 2v \\
= \frac{-4u^2 + 4yv}{y^2 - 4x} \]
PART II. Work Out Problems. (Questions 5 - 6)

5. (20 points) Let

\[ f(x, y) = y^2 - 4xy + \frac{4}{3}x^3 + 2. \]

Find all critical points and use the second partials test to classify each point as a relative maximum, a relative minimum or a saddle point.

\[ f_x = -4y + 4x^2 \quad f_y = 2y - 4x \]

Critical points

\[ f_y = 0 \iff y = 2x \]
\[ f_x = 0 \iff -y + x^2 = 0 \]
\[ \iff -2x + x^2 = 0 \]

\[ \begin{cases} x = 0, 2 \\ y = 0 \end{cases} \]
\[ \begin{cases} x = 2 \\ y = 4 \end{cases} \]
\[ (2, 4) \]

\[ D = \begin{vmatrix} 0 & -4 \\ -4 & 2 \end{vmatrix} = -16 < 0 \]

\[ f_{xx} = 8x \]
\[ f_{yy} = 2 \]
\[ f_{xy} = -4 \]

Saddle point

\[ D = \begin{vmatrix} 16 & -4 \\ -4 & 2 \end{vmatrix} = 16 > 0 \]

\[ f_{xx} = 16 > 0 \]

relative min
6. (20 points) Let \( f(x, y) = y^2 - x^2 \) and the domain \( D \) be the triangle ABC, where the vertices are \( A(-1, -2), B(2, 1) \) and \( C(-1, 1) \), that is, the domain \( D \) is the region bounded by \( x = -1 \), \( y = 1 \), and \( y = x - 1 \).

Find the absolute maximum and absolute minimum of \( f(x, y) \) over the domain \( D \).

\[
\begin{align*}
f_x &= -2x, \quad f_y = 2y \\
\text{Critical point: } (0, 0)
\end{align*}
\]

\[
\begin{align*}
g_{AC} &\quad x = -1 \\
f &= y^2 - 1 \leq f(y) = 2y, \quad -2 \leq y \leq 1 \\
P_1'(y) &= 2y, \quad P_1'(y) = 0 \text{ at } y = 0 \\
\Rightarrow \text{ point } (-1, 0)
\end{align*}
\]

\[
\begin{align*}
g_{BC} &\quad y = 1 \\
f &= 1 - x^2 = P_2(x), \quad -1 \leq x \leq 2 \\
P_2'(x) &= -2x \\
P_2'(x) = 0 \Rightarrow x = 0 \\
\Rightarrow \text{ point } (0, 1)
\end{align*}
\]

\[
\begin{align*}
g_{AB} &\quad y = x - 1 \\
f &= (x - 1)^2 - 1 = 2x + 1 = P_3(x) \\
-2 = P_3'(x) \neq 0 \\
\text{End points } A, B
\end{align*}
\]

Hence

\[
\begin{align*}
\text{Absolute max} = 3 \quad \text{(at A)} \\
\text{Absolute min} = -3 \quad \text{(at B)}
\end{align*}
\]