Solutions

MATH2360. SECTION 002. FALL 2010.

MIDTERM EXAMINATION 3

Name: ....

Signature: .... Date: ....

READ AND FOLLOW THESE INSTRUCTIONS

This booklet contains 4 pages, including this cover page. Check to see if any are missing. PRINT all the requested information above, and sign your name. Put your initials on the top of every page, in case the pages become separated. Books and notes **are not permissible.** Calculators are allowed. Do your work in the blank spaces and back of pages of this booklet.

There are 3 work-out problems making a total score of 100 points. Students should show all the work in order to receive full credits. Unsupported answers will receive little credit.

AFTER YOU FINISH THE EXAM, turn in the whole booklet.

Problem 1 (30)	Problem 2 (30)	Problem 3 (40)	Total Score (100)

1. (30 points) Let P_3 be the space of polynomials of degree less than 3. Let

$$E = \{1,\ x,\ 1 + x + x^2\} \text{ and } F = \{1 + x^2,\ 2 + x - x^2,\ 3 + 2x - 2x^2\}$$

be two sets of vectors in P_3.

(a) (10 points) Show that E and F are two bases of P_3.

Let $B_0 = \{1, x, x^2\}$ be the standard basis of P_3.

Let

$A =$ "$[E \to B_0]$" $= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ $\det A = 1 \neq 0$

$B =$ "$[F \to B_0]$" $= \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & -1 & -2 \end{pmatrix}$ $\det B = -1 \neq 0$

$\Rightarrow$ E, F are bases of P_3.

(b) (20 points) Find the transition matrix from F to E.

$$[F \to E] = [B_0 \to E][F \to B_0]$$
$$= A^{-1} B$$

$A^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

$\Rightarrow [F \to E] = \begin{pmatrix} 0 & 3 & 5 \\ -1 & 2 & 4 \\ 1 & -1 & -2 \end{pmatrix}$

2. (30 points) Let

$$A = \begin{pmatrix} 1 & 2 & 3 & -1 \\ 3 & 1 & -4 & 2 \\ -2 & 1 & 7 & -3 \end{pmatrix}.$$

(a) (10 points) Find the dimension and basis for the row space of A.

(b) (10 points) Find the dimension and basis for the column space of A.

(c) (10 points) Find the dimension and basis for the null space of A.

Using row operations

$$A \xrightarrow[(3)\to(3)+2(1)]{(2)\to(2)-3(1)} \begin{pmatrix} 1 & 2 & 3 & -1 \\ 0 & -5 & -13 & 5 \\ 0 & 5 & 13 & -5 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 13/5 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{(1)\to(1)-2(2)} \begin{pmatrix} \boxed{1} & 0 & -11/5 & 1 \\ 0 & \boxed{1} & 13/5 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \leftarrow \\ \leftarrow \\ \end{matrix}$$

$$\begin{matrix} \uparrow & \uparrow \end{matrix}$$

(a) Rank A = dimension of Row space $= 2$

basis of Row space $= \{(1, 0, -11/5, 1), (0, 1, 13/5, -1)\}$

(b) Dimension of column space = rank of $A = 2$

Using indices of first 2 columns of the echelon form, we infer basis of column sp

$= \left\{ \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\}$ (first 2 columns of A)

(c) Reduced echelon form above $\Longrightarrow$ dimension of null space is $4-2=2$

Basis (x_3, x_4 free)

$\left\{ \begin{pmatrix} 11/5 \\ -13/5 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

3. (40 points) Let L be a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ that is defined by

$$L(\mathbf{x}) = \begin{pmatrix} x_1 - 2x_2 \\ 2x_1 + x_2 - 3x_3 \\ -x_1 + 3x_2 + x_3 \end{pmatrix} \quad \text{for any } \mathbf{x} = (x_1, x_2, x_3)^T.$$

(a) (10 points) Find A, the standard matrix representation of L.

$$A = \begin{pmatrix} 1 & -2 & 0 \\ 2 & 1 & -3 \\ -1 & 3 & 1 \end{pmatrix}$$

(b) (20 points) Let $E = \{(0,1,0)^T,\ (1,0,1)^T,\ (1,1,0)^T\}$ be a basis of $\mathbb{R}^3$. $E = \{v_1, v_2, v_3\}$

Find B, the matrix representing L with respect to E, that is, $B = [L]_E = [L]_{E,E}$.

$$[L]_E = [L]_{E,E} = [E \to B_0]^{-1} A [E \to B_0] \quad (\ast)$$

where $[E \to B_0] = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ & $[E \to B_0]^{-1} = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}$

Calculations yield $[L]_E = \begin{pmatrix} 6 & -2 & 6 \\ 3 & 0 & 2 \\ -5 & 0 & -3 \end{pmatrix}$.

OR $[L]_E = [E \to B_0]^{-1} \big(\underbrace{L(0,1,0)}_{L(v_1)}\ \underbrace{L(1,0,1)}_{L(v_2)}\ \underbrace{L(1,1,0)}_{L(v_3)} \big)$

Row operation

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 1 & -2 & 1 & -1 \\ 1 & 0 & 1 & 1 & -1 & 3 \\ 0 & 1 & 0 & 3 & 0 & 2 \end{array}\right) \dashrightarrow \left(\begin{array}{c|ccc} I_3 & 6 & -2 & 6 \\ & 3 & 0 & 2 \\ & -5 & 1 & -3 \end{array}\right) \quad \boxed{\begin{matrix} 6 & -2 & 6 \\ 3 & 0 & 2 \\ -5 & 1 & -3 \end{matrix}} \to [L]_E$$

(c) (10 points) Is matrix A similar to matrix B? If that is the case, find a non-singular 3×3 matrix S such that $A = S^{-1} B S$.

From $(\ast)$ $\quad A = [E \to B_0] \underbrace{[L]_E}_{B} [E \to B_0]^{-1}$

hence $S = [E \to B_0]^{-1} = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}$