MATH2360. SECTION 002. FALL 2010.

MIDTERM EXAMINATION 3

Name: .................................................................

Signature: ..........................................................Date: ............................... 

READ AND FOLLOW THESE INSTRUCTIONS
This booklet contains 4 pages, including this cover page. Check to see if any are missing. PRINT all the requested information above, and sign your name. Put your initials on the top of every page, in case the pages become separated. Books and notes are not permissible. Calculators are allowed. Do your work in the blank spaces and back of pages of this booklet.

There are 3 work-out problems making a total score of 100 points. Students should show all the work in order to receive full credits. Unsupported answers will receive little credit.

AFTER YOU FINISH THE EXAM, turn in the whole booklet.

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1. (30 points) Let $P_3$ be the space of polynomials of degree less than 3. Let

$$E = \{1, x, 1 + x + x^2\}$$
and $F = \{1 + x^2, 2 + x - x^2, 3 + 2x - 2x^2\}$

be two sets of vectors in $P_3$.

(a) (10 points) Show that $E$ and $F$ are two bases of $P_3$.

Let $B_0 = \{1, x, x^2\}$ be the standard basis of $P_3$.

Let

$$A = \left[ \begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array} \right] \quad \text{det} \ A = 1 \neq 0$$

and

$$B = \left[ \begin{array}{ccc} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & -1 & -2 \end{array} \right] \quad \text{det} \ B = -1 \neq 0$$

implies $E, F$ are bases of $P_3$.

(b) (20 points) Find the transition matrix from $F$ to $E$.

$$\left[ \begin{array}{c} F \rightarrow E \end{array} \right] = \left[ \begin{array}{c} B_0 \rightarrow E \end{array} \right] \left[ \begin{array}{c} F \rightarrow B_0 \end{array} \right]$$

$$= A^{-1} B$$

$$A^{-1} = \left( \begin{array}{ccc} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{array} \right) \quad \Rightarrow \quad \left[ F \rightarrow E \right] = \left( \begin{array}{ccc} 0 & 3 & 5 \\ -1 & 2 & 4 \end{array} \right)$$
2. (30 points) Let

\[
A = \begin{pmatrix}
1 & 2 & 3 & -1 \\
3 & 1 & -4 & 2 \\
-2 & 1 & 7 & -3
\end{pmatrix}
\]

(a) (10 points) Find the dimension and basis for the row space of \( A \).

(b) (10 points) Find the dimension and basis for the column space of \( A \).

(c) (10 points) Find the dimension and basis for the null space of \( A \).

Using row operations

\[
A \xrightarrow{(1) \rightarrow (2) \rightarrow (3) - 2(1)} \begin{pmatrix}
1 & 2 & 3 & -1 \\
0 & -5 & -13 & 5 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\xrightarrow{(1) \rightarrow (1) \rightarrow (2)} \begin{pmatrix}
1 & 0 & 13/5 & -1 \\
0 & 1 & -11/5 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

(a) \( \text{rank } A = \text{ dimension of row space} = 2 \)

\( \text{basis of row space} = \{(1,1,1,1,1) , (0,1,13/5,1,1)\} \)

(b) \( \text{Dimension of column space} = \text{rank of } A = 2 \)

Using indices of first 2 columns of the echelon form, we infer basis of column space

\[
= \left\{ \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \quad \text{(first 2 columns of } A) \]

(c) \( \text{Reduced echelon form above} \rightarrow \text{dimension of null space} \)

\( 4 - 2 = 2 \)

\( \text{Basis} \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \right\} \)
3. (40 points) Let \( L \) be a linear transformation from \( \mathbb{R}^3 \) to \( \mathbb{R}^3 \) that is defined by
\[
L(x) = \begin{pmatrix}
    x_1 - 2x_2 \\
    2x_1 + x_2 - 3x_3 \\
    -x_1 + 3x_2 + x_3
\end{pmatrix}
\]
for any \( x = (x_1, x_2, x_3)^T \).

(a) (10 points) Find \( A \), the standard matrix representation of \( L \).
\[
A = \begin{pmatrix}
    1 & -2 & 0 \\
    2 & 1 & -3 \\
    -1 & 2 & 1
\end{pmatrix}
\]

(b) (20 points) Let \( E = \{ (0,1,0)^T, (1,0,1)^T, (1,1,0)^T \} \) be a basis of \( \mathbb{R}^3 \).
Find \( B \), the matrix representing \( L \) with respect to \( E \), that is, \( B = [L]_E = [L]_{E,E} \).
\[
[L]_E = [L]_{E,E} = [E\rightarrow B_0]^{-1} A [E\rightarrow B_0] \quad (\star)
\]
where \( [E\rightarrow B_0] = \begin{pmatrix}
    0 & 1 & 1 \\
    1 & 0 & 1 \\
    0 & 0 & 1
\end{pmatrix} \) and \( [E\rightarrow B_0]^{-1} = \begin{pmatrix}
    -1 & 1 & 1 \\
    0 & 1 & 1 \\
    0 & 0 & 1
\end{pmatrix} \)

Calculations yield
\[
[L]_E = \begin{pmatrix}
    6 & -2 & 6 \\
    3 & 0 & 2 \\
    -5 & 0 & -3
\end{pmatrix}
\]

OR
Row operation
\[
\begin{pmatrix}
    0 & 1 & 1 \\
    1 & 0 & 1 \\
    0 & 0 & 1
\end{pmatrix} \rightarrow \begin{pmatrix}
    1 & 0 & 0 \\
    0 & 1 & 0 \\
    0 & 0 & 1
\end{pmatrix}
\]

(c) (10 points) Is matrix \( A \) similar to matrix \( B \)? If that is the case, find a non-singular \( 3 \times 3 \) matrix \( S \) such that \( A = S^{-1} B S \).
From (\( \star \))
\[
A = [E\rightarrow B_0]^{-1} B [E\rightarrow B_0] = [E\rightarrow B_0]^{-1}
\]

hence \( S = [E\rightarrow B_0]^{-1} = \begin{pmatrix}
    -1 & 1 & 1 \\
    0 & 1 & 1 \\
    0 & 0 & 1
\end{pmatrix} \)