Conditional Probability and Independence

MATH 3342
Sections 2.4 and 2.5

Example

- What is the probability that you get an A on a test?
- Would you change your answer if you studied for 1+ hour/day for the week before the exam?
- How about if you did not?

Conditional Probability

- You have defined the events A and B.
- Suppose you are told that event B occurs.
- Given this information, what is the probability that event A occurs?

Definition

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

$$P(A) > 0$$

College Attendance

	City	Suburb	Country	Total
College	2,550	4,580	470	7,600
No College	3,450	920	430	4,800
Total	6,000	5,500	900	12,400

College Attendance

- If you choose a high school graduate at random, what is the probability that the person went to college?
- What is the conditional probability that you choose a college attendee given that the person lived in the suburbs?
- What is the conditional probability that you choose a person from the country given that they did **not** attend college?

The Multiplication Rule

- For any two events A and B,

$$P(A \cap B) = P(A \mid B)P(B)$$

Example: Dollar Value

- A = event that the value of the $ falls.
- B = event that the supplier demands changing the contract
- P(A) = 0.4
- P(B|A) = 0.8
- P(A and B) = ?

Some Facts

$$P(B \mid A) + P(B' \mid A) = 1$$

$$P(B \mid A)P(A) + P(B \mid A')P(A') = P(B)$$

The Law of Total Probability

- Let $A_1,\ldots,A_k$ be mutually exclusive and **exhaustive** events.
- Then for any other event B:

$$P(B) = \sum_{i=1}^{k} P(B \mid A_i)P(A_i)$$

Bayes' Rule

- Let $A_1,\ldots,A_k$ be mutually exclusive and **exhaustive** events with ***prior*** probabilities $P(A_i)$.
- Then for any other event B with P(B)>0, the ***posterior*** probability of A_j given B is:

$$P(A_j \mid B) = \frac{P(B \mid A_j)P(A_j)}{\sum_{i=1}^{k} P(B \mid A_i)P(A_i)}$$

Extension of $ Value Example

- Suppose we're also told that $P(B|A^C) = 0.3$
- What is P(A|B)?

Independence

- Events A and B are ***independent*** if

$$P(A | B) = P(A)$$

- Otherwise, they are ***dependent***.

Proposition

- A and B are independent if and only if

$$P(A \cap B) = P(A)P(B)$$

College Attendance

- Are college attendance and living in the suburbs independent?

College Attendance

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Example: Christmas Lights

- A string of Christmas lights contains 20 lights.
- If any light fails, the whole string fails.
- The probability that a light fails during a 3 year period is 0.02.
- The lights fail independently.
- What is the probability that the string stays lit for all 3 years?

Mutual Independence

- Events $A_1,\ldots,A_n$ are ***mutually independent*** if:
  - For every *k* (*k*=2,3,…,n) AND **every** subset of indices $i_1, i_2,\ldots,i_k$:

$$P(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}) = P(A_{i_1})P(A_{i_2})\cdots P(A_{i_k})$$