

VANISHING OF TATE HOMOLOGY AND DEPTH FORMULAS OVER LOCAL RINGS

LARS WINTHER CHRISTENSEN AND DAVID A. JORGENSEN

To Hans-Bjørn Foxby on his 65th birthday

ABSTRACT. Auslander's depth formula for pairs of Tor-independent modules over a regular local ring, $\text{depth}(M \otimes_R N) = \text{depth } M + \text{depth } N - \text{depth } R$, has been generalized in several directions; most significantly it has been shown to hold for pairs of Tor-independent modules over complete intersection rings.

In this paper we establish a depth formula that holds for every pair of Tate Tor-independent modules over a Gorenstein local ring. It subsumes previous generalizations of Auslander's formula and yields new results on vanishing of cohomology over certain Gorenstein rings.

INTRODUCTION

To infer properties, qualitative or quantitative, of a tensor product from properties of its factors is a delicate task. For finitely generated modules M and N over a commutative noetherian local ring R , Auslander [3] proved that the depth of the tensor product is given by the formula

$$(A) \quad \text{depth}_R(M \otimes_R N) = \text{depth}_R M + \text{depth}_R N - \text{depth } R,$$

provided that the projective dimension of M is finite and the two modules are Tor-independent, that is, the homology modules $\text{Tor}_i^R(M, N)$ vanish for $i \geq 1$. In particular, the equality (A) holds for every pair of finitely generated Tor-independent modules over a regular local ring.

The condition of finite projective dimension was first relaxed by Huneke and Wiegand [20], who established the validity of (A) for pairs of finitely generated Tor-independent modules over complete intersection local rings. Later, Araya and Yoshino [1] and Iyengar [23] showed that (A) holds for Tor-independent modules M and N , provided that M has finite complete intersection dimension.

In a different direction, Foxby [16] relaxed the condition of Tor-independence as follows. Let M and N be modules over a commutative noetherian local ring R and let P be a projective resolution of M . If M has finite projective dimension, then there is an equality,

$$(B) \quad \text{depth}_R(P \otimes_R N) = \text{depth}_R M + \text{depth}_R N - \text{depth } R.$$

Date: 21 November 2014.

2010 Mathematics Subject Classification. Primary 13D07. Secondary 13D02.

Key words and phrases. AB ring, complete intersection ring, depth formula, Gorenstein ring, rigidity of Tor, Tate homology.

This research was partly supported by NSA grants H98230-11-0214 (L.W.C.) and H98230-10-0197 (D.A.J.). Part of the work was done during the authors' visits to the University of Bielefeld (D.A.J) and the University of Paderborn (L.W.C.); the hospitality of both institutions is acknowledged with gratitude.

Implicit in this formula is an extension of the invariant “depth” to complexes of modules; we recall it in (1.3). The homology of the complex $P \otimes_R N$ is $\mathrm{Tor}_*^R(M, N)$, and if M and N are Tor-independent, then (B) reduces to Auslander’s formula (A).

* * *

The notion of Tate homology for modules over group algebras has a natural extension to modules over Gorenstein rings and, more generally, to modules of finite Gorenstein projective dimension over any ring. This theory was recently treated by Iacob [21]. We recall the basics in (2.2); a broader discussion is given in [13, sec. 2].

Let R be a commutative noetherian local ring. The central result in this paper, Theorem (2.3), establishes vanishing of Tate homology $\widehat{\mathrm{Tor}}_*^R(M, N)$ as a sufficient condition for the equality (B) to hold for a pair of R -modules (M, N) , where M has finite Gorenstein projective dimension. The Tate homology for such a pair vanishes if M has finite projective dimension over R , and if R is complete intersection, then Tor-independence implies vanishing of Tate homology. Thus the main theorem subsumes all of the aforementioned generalizations of Auslander’s depth formula. In fact, it goes further and subsumes several other generalizations obtained over the half-century that has passed since [3] appeared.

What is more significant, though, is that Theorem (2.3) applies to modules over all Gorenstein rings. The works of Iyengar and of Huneke and Wiegand established the depth formula in the realm of complete intersection rings. The depth is a cohomological invariant, and the cohomological behavior of modules over Gorenstein rings can stray dramatically from that of modules over complete intersection rings. Therefore, it came as a surprise to us that vanishing of Tate homology is sufficient to tame depth in this vastly wider context.

In Sections 4 and 5 we explore the consequences of the main theorem for modules over different classes of Gorenstein rings. For so-called AB rings, a notion coined by Huneke and Jorgensen [18], we obtain in Theorem (3.6) a precise bound on vanishing of cohomology of finitely generated modules. For modules over complete intersection rings, we obtain a derived depth formula for modules that satisfy an effectively verifiable condition on vanishing of Tate homology. That is, if $\widehat{\mathrm{Tor}}_i^R(M, N)$ vanishes for a finite number of consecutive indices—a number that only depends on R —then (B) holds; this is Theorem (4.6).

In the final section we prove a statement, dual to Theorem (2.3), for the width invariant and use it to establish a bound on vanishing of cohomology of finitely generated modules with vanishing Tate cohomology $\widehat{\mathrm{Ext}}_R^*(M, N)$.

1. DEPTH OF COMPLEXES

In this paper, R -complexes—that is, complexes of R -modules—are graded homologically. A complex

$$M : \quad \cdots \longrightarrow M_{i+1} \xrightarrow{\partial_{i+1}^M} M_i \xrightarrow{\partial_i^M} M_{i-1} \longrightarrow \cdots$$

is called *acyclic* if the homology complex $H(M)$ is the zero-complex. We use the notation $C_i(M)$ for the cokernel of the differential ∂_{i+1}^M . For $n \in \mathbb{Z}$ the n -fold shift of M is the complex $\Sigma^n M$ given by $(\Sigma^n M)_i = M_{i-n}$ and $\partial_i^{\Sigma^n M} = (-1)^n \partial_{i-n}^M$.

The notation $\sup M$ and $\inf M$ is used for the supremum and infimum of the set $\{i \in \mathbb{Z} \mid M_i \neq 0\}$, with the conventions $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$. A complex

M is called *bounded above* if $\sup M$ is finite, it is called *bounded below* if $\inf M$ is finite, and it is called *bounded* if it is bounded above and below.

For some of our proofs, we shall need the following variation on [5, lem. 4.4.F].

(1.1) **Lemma.** *Let F and M be R -complexes, and let P be a complex of finitely generated R -modules. If one of the following conditions holds*

- (a) F and M are bounded above, and P is bounded below, or
- (b) M and P are bounded

and P is a complex of projective R -modules or F is a complex of flat R -modules, then there is a natural isomorphism of R -complexes

$$\mathrm{Hom}_R(P, M) \otimes_R F \cong \mathrm{Hom}_R(P, M \otimes_R F).$$

Proof. For R -modules P , M , and F the tensor evaluation map

$$\omega_{PMF}: \mathrm{Hom}_R(P, M) \otimes_R F \rightarrow \mathrm{Hom}_R(P, M \otimes_R F)$$

given by

$$\omega_{PMF}(\psi \otimes f)(p) = \psi(p) \otimes f,$$

is a homomorphism. It is an isomorphism if P is finitely generated and projective, and also if P is finitely generated and F is flat; see [5, lem. 4.4.F].

Under either assumption, (a) or (b), the complex M is bounded above and P is bounded below. Assume, therefore, without loss of generality that one has $M_u = 0$ for all $u > 0$ and $P_u = 0$ for all $u < 0$. For every $n \in \mathbb{Z}$ one then has

$$\begin{aligned} (\mathrm{Hom}_R(P, M) \otimes_R F)_n &= \prod_{i \in \mathbb{Z}} \mathrm{Hom}_R(P, M)_i \otimes_R F_{n-i} \\ &= \prod_{i \in \mathbb{Z}} \left(\prod_{j \in \mathbb{Z}} \mathrm{Hom}_R(P_j, M_{j+i}) \right) \otimes_R F_{n-i} \\ &= \prod_{i \leq 0} \left(\bigoplus_{j=0}^{-i} \mathrm{Hom}_R(P_j, M_{j+i}) \right) \otimes_R F_{n-i} \end{aligned}$$

and

$$\begin{aligned} \mathrm{Hom}_R(P, M \otimes_R F)_n &= \prod_{j \in \mathbb{Z}} \mathrm{Hom}_R(P_j, (M \otimes_R F)_{j+n}) \\ &= \prod_{j \geq 0} \mathrm{Hom}_R(P_j, \prod_{h \in \mathbb{Z}} M_h \otimes_R F_{j+n-h}) \\ &= \prod_{j \geq 0} \mathrm{Hom}_R(P_j, \prod_{i \leq -j} M_{j+i} \otimes_R F_{n-i}). \end{aligned}$$

If (a) holds, then one can assume that F_u is zero for all $u > 0$, whence

$$(\mathrm{Hom}_R(P, M) \otimes_R F)_n = \bigoplus_{i=n}^0 \bigoplus_{j=0}^{-i} \mathrm{Hom}_R(P_j, M_{j+i}) \otimes_R F_{n-i}$$

and

$$\begin{aligned} \mathrm{Hom}_R(P, M \otimes_R F)_n &= \bigoplus_{j=0}^{-n} \mathrm{Hom}_R(P_j, \bigoplus_{i=n}^{-j} M_{j+i} \otimes_R F_{n-i}) \\ &= \bigoplus_{i=n}^0 \bigoplus_{j=0}^{-i} \mathrm{Hom}_R(P_j, M_{j+i} \otimes_R F_{n-i}). \end{aligned}$$

The map from $\mathrm{Hom}_R(P, M \otimes_R F)$ to $\mathrm{Hom}_R(P, M) \otimes_R F$ with degree n component $\bigoplus_{i=n}^0 \bigoplus_{j=0}^{-i} (-1)^{j(n-i)} \omega_{P_j M_{j+i} F_{n-i}}$ is a morphism of complexes; this is elementary

to verify. It follows that it is an isomorphism if the modules in P are projective or the modules in F are flat.

If (b) holds, then a similar argument applies. \square

(1.2) **Resolutions.** A morphism of R -complexes that induces an isomorphism in homology is called a *quasi-isomorphism* and indicated by the symbol ' \simeq '.

An R -complex P is called *semi-projective* if each module P_i is projective, and the functor $\mathrm{Hom}_R(P, -)$ preserves quasi-isomorphisms. Every bounded below complex of projective R -modules is semi-projective. Similarly, an R -complex I is called *semi-injective* if each module I_i is injective, and the functor $\mathrm{Hom}_R(-, I)$ preserves quasi-isomorphisms. Every bounded above complex of injective R -modules is semi-injective. The following facts are proved in [6].

- (P) Every R -complex M has a semi-projective resolution. That is, there is a quasi-isomorphism $\pi: P \rightarrow M$, where P is a semi-projective complex with $P_i = 0$ for all $i < \inf M$. Moreover, if $\mathrm{H}(M)$ is bounded below, then M has a semi-projective resolution $P' \xrightarrow{\simeq} M$ with $P'_i = 0$ for all $i < \inf \mathrm{H}(M)$.
- (I) Every R -complex M has a semi-injective resolution. That is, there is a quasi-isomorphism $\iota: M \rightarrow I$, where I is semi-injective with $I_i = 0$ for all $i > \sup M$. Moreover, if $\mathrm{H}(M)$ is bounded above, then M has a semi-injective resolution $M \xrightarrow{\simeq} I'$ with $I'_i = 0$ for all $i > \sup \mathrm{H}(M)$.

For an R -module M , a semi-projective (-injective) resolution is just a projective (injective) resolution in the classic sense; see [8].

We use the standard notations $- \otimes_R^{\mathbf{L}} -$ and $\mathbf{R}\mathrm{Hom}_R(-, -)$ for the derived tensor product and derived Hom of complexes; they are computed by way of the resolutions described above. Extending the usual definitions of Tor and Ext for modules, set

$$\mathrm{Tor}_i^R(M, N) = \mathrm{H}_i(M \otimes_R^{\mathbf{L}} N) \quad \text{and} \quad \mathrm{Ext}_R^i(M, N) = \mathrm{H}_{-i}(\mathbf{R}\mathrm{Hom}_R(M, N))$$

for R -complexes M and N and $i \in \mathbb{Z}$. In another extension of classic notions, define the projective and injective dimension of an R -complex by

$$\mathrm{pd}_R M = \inf\{\sup P \mid P \xrightarrow{\simeq} M \text{ is a semi-projective resolution}\}$$

and

$$\mathrm{id}_R M = \inf\{-\inf I \mid M \xrightarrow{\simeq} I \text{ is a semi-injective resolution}\}.$$

Setup. From this point, R denotes a local ring with maximal ideal \mathfrak{m} and residue field $k = R/\mathfrak{m}$. The *embedding dimension* of R , written $\mathrm{edim} R$, is the minimal number of generators of \mathfrak{m} . The *codepth* and *codimension* of R are the differences

$$\mathrm{codepth} R = \mathrm{edim} R - \mathrm{depth} R \quad \text{and} \quad \mathrm{codim} R = \mathrm{edim} R - \dim R,$$

where $\dim R$ denotes the Krull dimension of R .

The depth of an R -complex is defined by extension of the homological characterization of depth of finitely generated modules.

(1.3) **Depth.** Let M be an R -complex. The *depth* of M is defined as

$$\mathrm{depth}_R M = -\sup \mathrm{H}(\mathbf{R}\mathrm{Hom}_R(k, M)).$$

If $\mathrm{H}(M)$ is bounded above, then M has a semi-injective resolution $M \xrightarrow{\simeq} I$ with $I_i = 0$ for $i > \sup \mathrm{H}(M)$; see (1.2)(I). Thus, for every R -complex M one has

$$(1.3.1) \quad \mathrm{depth}_R M \geq -\sup \mathrm{H}(M).$$

(1.4) **The derived depth formula.** Let M and N be R -complexes. We say that *the derived depth formula holds for M and N* if there is an equality

$$(1.4.1) \quad \text{depth}_R(M \otimes_R^{\mathbf{L}} N) = \text{depth}_R M + \text{depth}_R N - \text{depth } R.$$

Note that this is just a rewrite of the equality (B) in the introduction. By [16, lem. (2.1)] the derived depth formula holds for complexes M and N if M has finite projective dimension and $H(N)$ is bounded above.

The next result is due to Dwyer and Greenlees [15, 6.5] and to Foxby and Iyengar [17, 2.3 and 4.1].

(1.5) **Proposition.** *Let K be the Koszul complex on a set of generators for \mathfrak{m} . For an R -complex M , the following conditions are equivalent.*

- (i) $H(k \otimes_R^{\mathbf{L}} M) = 0$;
- (ii) $H(K \otimes_R M) = 0$;
- (iii) $H(\text{Hom}_R(K, M)) = 0$;
- (iv) $H(\mathbf{R}\text{Hom}_R(k, M)) = 0$. □

2. DEPTH AND VANISHING OF TATE HOMOLOGY—THE MAIN THEOREM

We start by recalling some facts from [13] and [21].

(2.1) **Complete resolutions.** An acyclic complex T of projective R -modules is called *totally acyclic*, if the complex $\text{Hom}_R(T, Q)$ is acyclic for every projective R -module Q . An R -module G is called *Gorenstein projective* if there exists such a totally acyclic complex T with $C_0(T) \cong G$.

Let M be an R -complex. A *complete (projective) resolution* of M is a diagram

$$(2.1.1) \quad T \xrightarrow{\tau} P \xrightarrow{\pi} M,$$

where π is a semi-projective resolution, T is a totally acyclic complex of projective R -modules, and τ_i is an isomorphism for $i \gg 0$. The *Gorenstein projective dimension* of M , written $\text{Gpd}_R M$, is the least integer n such that there exists a complete resolution (2.1.1) where τ_i is an isomorphism for all $i \geq n$. In particular, $\text{Gpd}_R M$ is finite if and only if M has a complete resolution. Notice that the homology $H(M)$ is bounded above if $\text{Gpd}_R M$ is finite. Note also that a complex of finite projective dimension has finite Gorenstein projective dimension; indeed, $0 \rightarrow P \rightarrow M$ is a complete resolution for every semi-projective resolution $P \rightarrow M$ with P bounded above.

(2.2) **Tate homology.** Let M be an R -complex of finite Gorenstein projective dimension, and let $T \rightarrow P \rightarrow M$ be a complete resolution. For an R -complex N , the Tate homology of M with coefficients in N is defined as

$$\widehat{\text{Tor}}_i^R(M, N) = H_i(T \otimes_R N).$$

This definition is independent of the choice of complete resolution; see [21] or [13, sec. 2] for details. In particular, one has

$$(2.2.1) \quad \widehat{\text{Tor}}_i^R(M, N) \cong \text{Tor}_i^R(M, N) \quad \text{for } i > \text{Gpd}_R M + \sup N.$$

If M has finite projective dimension or if N is bounded above and of finite projective dimension, then $\widehat{\text{Tor}}_i^R(M, N) = 0$ for all $i \in \mathbb{Z}$; see [13, prop. (2.5) and lem. (2.7)].

The next theorem is our central result. For an R -complex M of finite projective dimension one has $\widehat{\mathrm{Tor}}_*^R(M, -) = 0$, so the theorem subsumes [16, lem. (2.1)]. The boundedness condition on the complex N , as opposed to its homology, reflects the fact that Tate homology is not a functor on the derived category over R ; see the remarks before [13, prop. (2.5)].

(2.3) **Theorem.** *Let M be an R -complex of finite Gorenstein projective dimension and let N be a bounded above R -complex. If one has $\widehat{\mathrm{Tor}}_i^R(M, N) = 0$ for all $i \in \mathbb{Z}$, then the derived depth formula holds for M and N . That is, one has*

$$\mathrm{depth}_R(M \otimes_R^{\mathbf{L}} N) = \mathrm{depth}_R M + \mathrm{depth}_R N - \mathrm{depth} R.$$

Note that the homology complex $\mathrm{H}(M \otimes_R^{\mathbf{L}} N)$ is bounded above by (2.2.1).

Proof. Choose a complete resolution $T \xrightarrow{\tau} P \rightarrow M$ and let $\pi': P' \xrightarrow{\simeq} N$ be a semi-projective resolution. The quasi-isomorphism $P \otimes_R \pi': P \otimes_R P' \xrightarrow{\simeq} P \otimes_R N$ is then a semi-projective resolution, and the Künneth formula yields

$$\begin{aligned} \mathrm{H}((M \otimes_R^{\mathbf{L}} N) \otimes_R^{\mathbf{L}} k) &\cong \mathrm{H}((P \otimes_R P') \otimes_R k) \\ &\cong \mathrm{H}((P \otimes_R k) \otimes_k (P' \otimes_R k)) \\ &\cong \mathrm{H}(M \otimes_R^{\mathbf{L}} k) \otimes_k \mathrm{H}(N \otimes_R^{\mathbf{L}} k). \end{aligned}$$

It now follows from (1.3.1) and Proposition (1.5) that $\mathrm{depth}_R(M \otimes_R^{\mathbf{L}} N)$ is finite if and only if $\mathrm{depth}_R M$ and $\mathrm{depth}_R N$ are both finite. In particular, the left- and right-hand sides of the equality we aim to prove are simultaneously finite.

Assume that both M and N have finite depth. Consider the degreewise split exact sequence of R -complexes $0 \rightarrow P \rightarrow \mathrm{Cone} \tau \rightarrow \Sigma T \rightarrow 0$ and apply the functor $-\otimes_R N$ to it. By assumption, the complex $(\Sigma T) \otimes_R N \cong \Sigma(T \otimes_R N)$ is acyclic, so there is a quasi-isomorphism $P \otimes_R N \xrightarrow{\simeq} (\mathrm{Cone} \tau) \otimes_R N$. In high degrees $K = \mathrm{Cone} \tau$ is isomorphic to the mapping cone of an isomorphism, and the mapping cone of an isomorphism is contractible. Therefore there exist homomorphisms $\sigma_i: K_i \rightarrow K_{i+1}$, such that one has $1^{K_i} = \sigma_{i-1} \partial_i^K + \partial_{i+1}^K \sigma_i$ for $i \gg 0$. Since K is a complex of projective modules, and $\partial_{i+1}^K \sigma_i = 1^{\mathrm{Im} \partial_{i+1}^K}$ holds for $i \gg 0$, it follows that the modules $\mathrm{Ker} \partial_i^K = \mathrm{Im} \partial_{i+1}^K \cong C_{i+1}(K)$ are projective for $i \gg 0$. Fix $n \gg 0$ and consider the contractible subcomplex $J = \cdots \rightarrow K_{n+2} \rightarrow K_{n+1} \rightarrow \mathrm{Im} \partial_{n+1}^K \rightarrow 0$. The sequence $0 \rightarrow J \rightarrow K \rightarrow K/J \rightarrow 0$ is split exact, because the quotient complex $L = K/J = 0 \rightarrow C_n(K) \rightarrow K_{n-1} \rightarrow \cdots$ consists of projective modules. The complex $J \otimes_R N$ is contractible, so there are quasi-isomorphisms,

$$(1) \quad P \otimes_R N \xrightarrow{\simeq} K \otimes_R N \xrightarrow{\simeq} L \otimes_R N.$$

Choose a semi-injective resolution $\iota: N \xrightarrow{\simeq} I$, where ι is injective and I is bounded above; see (1.2)(I). Consider the exact sequence $0 \rightarrow N \xrightarrow{\iota} I \rightarrow C \rightarrow 0$ of R -complexes. The complex $C = \mathrm{Coker} \iota$ is bounded above and acyclic, and hence so is the complex $L \otimes_R C$; cf. [11, lem. 2.13]. Thus, there is a quasi-isomorphism

$$(2) \quad L \otimes_R N \xrightarrow{\simeq} L \otimes_R I.$$

The complex $L \otimes_R I$ is bounded above and consists of injective R -modules, so it follows from (1), (2), and [5, 1.4.I] that it is a semi-injective resolution of the

complex $P \otimes_R N \simeq M \otimes_R^{\mathbf{L}} N$. The third equality in the next computation follows from Lemma (1.1).

$$\begin{aligned}
 \text{depth}_R(M \otimes_R^{\mathbf{L}} N) &= -\sup \mathbf{H}(\mathbf{R}\text{Hom}_R(k, M \otimes_R^{\mathbf{L}} N)) \\
 &= -\sup \mathbf{H}(\text{Hom}_R(k, I \otimes_R L)) \\
 &= -\sup \mathbf{H}(\text{Hom}_R(k, I) \otimes_R L) \\
 &= -\sup \mathbf{H}(\text{Hom}_R(k, I) \otimes_k k \otimes_R L) \\
 &= -\sup \mathbf{H}(\text{Hom}_R(k, I)) - \sup \mathbf{H}(k \otimes_R L) \\
 &= \text{depth}_R N - \sup \mathbf{H}(k \otimes_R L)
 \end{aligned}$$

For $N = R$ this equality reads

$$\text{depth}_R M = \text{depth } R - \sup \mathbf{H}(k \otimes_R L).$$

The desired equality follows by elimination of the quantity $\sup \mathbf{H}(k \otimes_R L)$. \square

(2.4) **Example.** Let M be an R -module of finite Gorenstein projective dimension.

(1) If N is an R -module of finite injective dimension, then $\widehat{\text{Tor}}_*^R(M, N) = 0$ holds by [11, lem. 2.3 and prop. 3.9], so the derived depth formula holds for M and N .

(2) If N is an R -module of finite projective dimension, and $\iota: N \rightarrow I$ is an injective preenvelope, then the derived depth formula holds for M and $N' = \text{Coker } \iota$, as $\widehat{\text{Tor}}_*^R(M, N) = 0$ and $\widehat{\text{Tor}}_*^R(M, I) = 0$ force $\widehat{\text{Tor}}_*^R(M, N') = 0$.

(3) If N is an R -module of finite injective dimension, and $\pi: P \rightarrow N$ is a projective precover, then the derived depth formula holds for M and $\text{Ker } \pi$.

Part (1) is known from [12, thm. 6.3], while (2) and (3) appear to be new.

(2.5) **Gorenstein rings.** Let R be Gorenstein. Every R -complex with bounded above homology has finite Gorenstein projective dimension; see [30, thm. 3.11]. Thus by Theorem (2.3) the derived depth formula holds for R -complexes M and N with $\mathbf{H}(M)$ and N bounded above and $\widehat{\text{Tor}}_*^R(M, N) = 0$.

For finitely generated modules, the Gorenstein projective dimension coincides with Auslander and Bridger's notion of G-dimension; see [11, prop. 3.8]. The following equality is known as the Auslander–Bridger Formula; it holds for every finitely generated module M of finite G-dimension,

$$(2.5.1) \quad \text{G-dim}_R M = \text{depth } R - \text{depth}_R M.$$

For finitely generated R -modules with $\widehat{\text{Tor}}_*^R(M, N) = 0$ there is hence an equality

$$(2.5.2) \quad \text{G-dim}_R(M \otimes_R^{\mathbf{L}} N) = \text{G-dim}_R M + \text{G-dim}_R N,$$

which represents a natural generalization of [3, cor. 1.3] to G-dimension.

(2.6) **Remark.** In [26] Jorgensen and Şega give an example of an artinian Gorenstein ring R and a finitely generated R -module M , such that for every $s \geq 0$ there exists a finitely generated R -module N_s with

$$s = \sup \mathbf{H}(M \otimes_R^{\mathbf{L}} N_s) = -\text{depth}_R(M \otimes_R^{\mathbf{L}} N_s).$$

Thus, over a Gorenstein ring, boundedness of $\mathbf{H}(M \otimes_R^{\mathbf{L}} N)$ —that is, vanishing of $\text{Tor}_{\geq 0}^R(M, N)$ —does not *per se* guarantee that the derived depth formula holds. This phenomenon disappears over so-called AB rings, where vanishing of homology is easier to control.

3. AB RINGS

Recall from Huneke and Jorgensen [18] that a local ring R is called *AB* if it is Gorenstein, and the following holds for all finitely generated R -modules M and N ,

$$\mathrm{Ext}_R^i(M, N) = 0 \text{ for } i \gg 0 \text{ implies } \mathrm{Ext}_R^i(M, N) = 0 \text{ for } i > \dim R.$$

At the end of this section we apply our main theorem to provide a precise bound for the vanishing of cohomology $\mathrm{Ext}_R^i(M, N)$ for modules over AB rings; it turns out to depend only on M .

(3.1) Tate cohomology. Let M be an R -complex of finite Gorenstein projective dimension, and let $T \rightarrow P \rightarrow M$ be a complete resolution. For an R -complex N , the Tate cohomology of M with coefficients in N is defined as

$$\widehat{\mathrm{Ext}}_R^i(M, N) = \mathrm{H}_{-i}(\mathrm{Hom}_R(T, N)).$$

This definition is independent of the choice of complete resolution; see [30] for details. In particular, one has

$$(3.1.1) \quad \widehat{\mathrm{Ext}}_R^i(M, N) \cong \mathrm{Ext}_R^i(M, N) \quad \text{for } i > \mathrm{Gpd}_R M - \inf N.$$

If M has finite projective dimension or if N is bounded below and of finite injective dimension, then one has $\widehat{\mathrm{Ext}}_R^i(M, N) = 0$ for all $i \in \mathbb{Z}$; see [30, thm. 4.5] and [13, lem. (4.2)].

In some sense, the conditions in the main theorem (2.3) are easier to verify for modules over AB rings. Not only is finiteness of Gorenstein projective dimension automatic, per the next lemma one only needs vanishing of homology in high degrees. For modules over a familiar class of AB rings, namely complete intersections, it becomes truly easier, as one only needs vanishing of a finite number of homology modules; see Theorem (4.6).

(3.2) Proposition. *Let R be AB and let M and N be finitely generated R -modules. The following assertions hold.*

- (a) $\mathrm{Tor}_i^R(M, N) = 0$ for $i \gg 0$ implies $\widehat{\mathrm{Tor}}_i^R(M, N) = 0$ for all $i \in \mathbb{Z}$.
- (b) $\mathrm{Ext}_R^i(M, N) = 0$ for $i \gg 0$ implies $\widehat{\mathrm{Ext}}_R^i(M, N) = 0$ for all $i \in \mathbb{Z}$.

Proof. As R is Gorenstein, M has finite G-dimension. Let $T \rightarrow P \rightarrow M$ be a complete resolution of M . For all integers i and n with $i > n$ one has

$$(1) \quad \widehat{\mathrm{Tor}}_i^R(M, N) = \mathrm{H}_i(T \otimes_R N) = \mathrm{Tor}_{i-n}^R(C_n(T), N).$$

The Krull dimension $d = \dim R$ is an upper bound for the G-dimension of a finitely generated R -module, cf. (2.5.1). It follows from (2.2.1) that there are isomorphisms $\widehat{\mathrm{Tor}}_i^R(M, N) \cong \mathrm{Tor}_i^R(M, N)$ for $i > d$. Thus, if the homology modules $\mathrm{Tor}_i^R(M, N)$ vanish for $i \gg 0$, then so do the modules $\widehat{\mathrm{Tor}}_i^R(M, N)$. For every $n \in \mathbb{Z}$ it follows that the modules $\mathrm{Tor}_j^R(C_n(T), N)$ vanish for $j \gg 0$, and since R is AB they vanish for $j > 0$; see [18, thm. 3.4]. Now it follows from (1) that all the Tate homology modules $\widehat{\mathrm{Tor}}_i^R(M, N)$ vanish. This proves part (a); the proof of (b) is similar. \square

(3.3) Remark. If R is AB and M and N are finitely generated R -modules with $\mathrm{Tor}_i^R(M, N) = 0$ for $i \gg 0$, then it follows from (2.5) and Proposition (3.2)(a) that the derived depth formula holds for M and N , and hence that (2.5.2) holds.

(3.4) **Remark.** Let R be AB and let M and N be finitely generated Gorenstein projective R -modules with (minimal) complete resolutions T and A . If one has $\mathrm{Tor}_i^R(M, N) = 0$ for $i \gg 0$, then it follows from Proposition (3.2)(b) and [13, cor. (6.2)] that the module $M \otimes_R N$ is Gorenstein projective with (minimal) complete resolution $T \otimes_R^{\mathfrak{M}} A$. Here $T \otimes_R^{\mathfrak{M}} A$ denotes the pinched tensor product defined in [13].

Recall that R is *complete intersection* if there exists a surjective ring homomorphism $\pi: Q \rightarrow \widehat{R}$, where Q is a complete regular local ring, and $\mathrm{Ker} \pi$ is generated by a Q -regular sequence. The least length of such a sequence equals $\mathrm{codim} R$. Complete intersection rings are perhaps the best known examples of AB rings, and they are studied further in the next section. Here we mention, in passing, a result of Celikbas and Dao [9, cor. 1.3] that constitutes a partial converse to Remark (3.3).

(3.5) Let R be complete intersection of codimension c , and assume that $R_{\mathfrak{p}}$ is regular for every prime ideal \mathfrak{p} in R with $\mathrm{ht}_R \mathfrak{p} \leq c$. If the tensor product $M \otimes_R N$ of two finitely generated Gorenstein projective R -modules is Gorenstein projective, then one has $\mathrm{Tor}_i^R(M, N) = 0$ for all $i \geq 1$.

The next result on vanishing of cohomology was established for complete intersection rings by Araya and Yoshino [1, thm. 4.2].

(3.6) **Theorem.** *Assume that R is AB and let M and N be finitely generated R -modules. If one has $\mathrm{Ext}_R^i(M, N) = 0$ for $i \gg 0$, then the next equality holds,*

$$\sup\{i \in \mathbb{Z} \mid \mathrm{Ext}_R^i(M, N) \neq 0\} = \mathrm{depth} R - \mathrm{depth}_R M.$$

Proof. As R is Gorenstein, the module M has finite G-dimension. Choose a complete resolution $T \xrightarrow{\tau} P \rightarrow M$. By Proposition (3.2)(b) the Tate cohomology modules $\widehat{\mathrm{Ext}}_R^i(M, N)$ vanish for all i , so the complex $\mathrm{Hom}_R(T, N)$ is acyclic. Let E be the injective hull of the residue field. The complex

$$\mathrm{Hom}_R(\mathrm{Hom}_R(T, N), E) \cong T \otimes_R \mathrm{Hom}_R(N, E),$$

is then acyclic as well; the isomorphism is homomorphism evaluation [5, 4.4.I] in each degree. It follows that the Tate homology modules $\widehat{\mathrm{Tor}}_i^R(M, \mathrm{Hom}_R(N, E))$ vanish for all i , so Theorem (2.3) applies to the modules M and $\mathrm{Hom}_R(N, E)$. The latter module has depth 0, so one has

$$(1) \quad \mathrm{depth}_R(M \otimes_R^{\mathbf{L}} \mathrm{Hom}_R(N, E)) = \mathrm{depth}_R M - \mathrm{depth} R.$$

The module $\mathrm{Hom}_R(N, E)$ is only supported on the maximal ideal of R , and so are the homology modules of the complex $M \otimes_R^{\mathbf{L}} \mathrm{Hom}_R(N, E) \simeq P \otimes_R \mathrm{Hom}_R(N, E)$. This explains the first equality in the next chain; the second equality is homomorphism evaluation.

$$(2) \quad \begin{aligned} -\mathrm{depth}_R(P \otimes_R \mathrm{Hom}_R(N, E)) &= \sup \mathrm{H}(P \otimes_R \mathrm{Hom}_R(N, E)) \\ &= \sup \mathrm{H}(\mathrm{Hom}_R(\mathrm{Hom}_R(P, N), E)) \\ &= -\inf \mathrm{H}(\mathrm{Hom}_R(P, N)) \\ &= -\inf \mathrm{H}(\mathbf{R}\mathrm{Hom}_R(M, N)) \\ &= \sup\{i \in \mathbb{Z} \mid \mathrm{Ext}_R^i(M, N) \neq 0\} \end{aligned}$$

The desired equality now follows from (1) and (2). \square

4. COMPLETE INTERSECTIONS

Homology of finitely generated modules over complete intersection rings is rigid in the following sense; see [19, remarks before thm. 1.9].

(4.1) **Fact.** Let R be complete intersection of codimension c and let M and N be finitely generated R -modules. Let $n \geq 0$ be an integer; if one has $\mathrm{Tor}_i^R(M, N) = 0$ for all i with $n + c \geq i \geq n$, then one has $\mathrm{Tor}_i^R(M, N) = 0$ for all $i \geq n$.

Combined with Theorem (2.3) and [4, thm. 4.9] this fact has the following consequence.

(4.2) **Corollary.** Let R be complete intersection of codimension c , and let M and N be finitely generated R -modules. If one has $\mathrm{Tor}_i^R(M, N) = 0$ for $c+1$ consecutive values of $i \geq 0$, then the Tate homology modules $\widehat{\mathrm{Tor}}_i^R(M, N)$ vanish for all $i \in \mathbb{Z}$, and the derived depth formula holds for M and N . \square

One goal of this section is to obtain a similar result for modules that are not finitely generated; see Theorem (4.6).

Recall from [7] that a (*codimension c*) *quasi-deformation* of R is a diagram of local homomorphism $R \xrightarrow{\rho} R' \xleftarrow{\pi} Q$, where ρ is flat, and π is surjective with kernel generated by a Q -regular sequence (of length c).

(4.3) **Lemma.** Let M and N be R -complexes with N bounded above. If there exists a codimension c quasi-deformation $R \rightarrow R' \leftarrow Q$ such that $\mathrm{pd}_Q(R' \otimes_R M)$ is finite, then $\mathrm{Gpd}_R M$ is finite, and the following conditions are equivalent.

- (i) $\widehat{\mathrm{Tor}}_i^R(M, N) = 0$ for all $i \in \mathbb{Z}$.
- (ii) $\mathrm{Tor}_i^R(M, N) = 0$ for all $i \gg 0$.
- (ii') $\widehat{\mathrm{Tor}}_i^R(M, N) = 0$ for all $i \ll 0$.
- (iii) $\mathrm{Tor}_i^R(M, N) = 0$ for $c+1$ consecutive values of $i > \mathrm{Gpd}_R M + \sup N$.
- (iv) $\widehat{\mathrm{Tor}}_i^R(M, N) = 0$ for $c+1$ consecutive values of i .

To not interrupt the flow, we defer the proof of this lemma to the end of the section.

We say that an R -complex M has *finite CI-dimension*, if there is a quasi-deformation of R such that $\mathrm{pd}_Q(R' \otimes_R M)$ is finite. For modules M and N with $\mathrm{Tor}_{\geq 1}^R(M, N) = 0$ the next theorem recovers Iyengar's [23, thm. 4.3]; it also subsumes a recent generalization of this result due to Sahandi, Sharif, and Yassemi [27, thm. 3.3].

(4.4) **Theorem.** Let M and N be R -complexes with $\mathrm{H}(N)$ bounded above. If M has finite CI-dimension, and one of the following conditions holds.

- (a) one has $\mathrm{Tor}_i^R(M, N) = 0$ for $i \gg 0$; or
- (b) the complex N is bounded above, and one has $\widehat{\mathrm{Tor}}_i^R(M, N) = 0$ for $i \ll 0$;

then the Tate homology modules $\widehat{\mathrm{Tor}}_i^R(M, N)$ vanish for all $i \in \mathbb{Z}$, and the derived depth formula holds for M and N .

We precede the proof with a technical observation.

(4.5) **Remark.** Let N be an R -complex with $\mathrm{H}(N)$ bounded above. Set $s = \sup \mathrm{H}(N)$ and let N' be the soft truncation of N at s , i.e. the bounded above

complex $0 \rightarrow C_s(N) \rightarrow N_{s-1} \rightarrow \cdots$. The natural morphism $N \twoheadrightarrow N'$ is a quasi-isomorphism; in particular, one has $\text{depth}_R N' = \text{depth}_R N$. For every R -complex M there are isomorphisms

$$\text{Tor}_i^R(M, N') \cong \text{Tor}_i^R(M, N) \text{ for all } i \in \mathbb{Z}.$$

Moreover, one has $\text{depth}_R(M \otimes_R^{\mathbf{L}} N') = \text{depth}_R(M \otimes_R^{\mathbf{L}} N)$, so the depth formula holds for M and N if and only if it holds for M and N' .

Proof of Theorem (4.4). Under the assumption that N is bounded above and $\widehat{\text{Tor}}_i^R(M, N) = 0$ holds for $i \ll 0$, the assertions follow immediately from Lemma (4.3) and Theorem (2.3).

Assume now that $\text{Tor}_i^R(M, N) = 0$ holds for $i \gg 0$. By Remark (4.5) we can replace N by its soft truncation at $\text{sup H}(N)$; that is, we can assume that N is bounded above. Now the assertions follow as above. \square

Over a complete intersection ring, vanishing of Tate homology $\widehat{\text{Tor}}_i^R(M, N)$ for all $i \in \mathbb{Z}$ can be inferred from a finite gap in (Tate) homology, and the length of that gap is independent of M and N .

(4.6) Theorem. *Let R be complete intersection of codimension c and dimension d . Let M and N be R -complexes with $\text{H}(M)$ and $\text{H}(N)$ bounded above. If one of the following conditions holds,*

- (a) $\text{Tor}_i^R(M, N) = 0$ for $c+1$ consecutive values of $i > d + \text{sup H}(M) + \text{sup H}(N)$; or
- (b) the complex N is bounded above, and one has $\widehat{\text{Tor}}_i^R(M, N) = 0$ for $c+1$ consecutive values of i ;

then the Tate homology modules $\widehat{\text{Tor}}_i^R(M, N)$ vanish for all $i \in \mathbb{Z}$, and the derived depth formula holds for M and N .

Proof. A complete intersection ring is Gorenstein, so by [30, thm. 3.10] one has

$$\text{Gpd}_R M \leq d + \text{sup H}(M) < \infty.$$

By assumption there is a homomorphism $\pi: Q \twoheadrightarrow \widehat{R}$, where Q is a complete regular local ring, and $\text{Ker } \pi$ is generated by a Q -regular sequence x_1, \dots, x_c . Thus, the diagram $R \rightarrow \widehat{R} \leftarrow Q$ is a codimension c quasi-deformation of R , and because $\text{H}(\widehat{R} \otimes_R M)$ is bounded above, one has $\text{pd}_Q(\widehat{R} \otimes_R M) < \infty$.

If N is bounded above and one has $\widehat{\text{Tor}}_i^R(M, N) = 0$ for $c+1$ consecutive values of i , then Lemma (4.3) yields $\widehat{\text{Tor}}_i^R(M, N) = 0$ for all $i \in \mathbb{Z}$. As $\text{Gpd}_R M$ is finite, the derived depth formula holds for M and N by Theorem (2.3).

Now, set $s = \text{sup H}(N)$ and assume that one has $\text{Tor}_i^R(M, N) = 0$ for $c+1$ consecutive values of $i > d + \text{sup H}(M) + s$. By Remark (4.5) we can replace N by its soft truncation at s ; that is, we can assume that N is bounded above with $\text{sup } N = s$. Now the assertions follow as above. \square

(4.7) Remark. Over a Gorenstein ring R of dimension d , the number $d + \text{sup H}(M)$ is an upper bound for the Gorenstein projective dimension of every R -complex M . The proof above and the fact that Tate homology is balanced, see [13, sec. 5], shows that one can replace the quantity $d + \text{sup H}(M) + \text{sup H}(N)$ in the theorem with $\min\{\text{Gpd}_R M + \text{sup H}(N), \text{Gpd}_R N + \text{sup H}(M)\}$.

For complexes with bounded and degreewise finitely generated homology, for finitely generated modules in particular, the CI-dimension agrees with Avramov, Gasharov, and Peeva's notion of CI-dimension; see [7, 28].

(4.8) **Corollary.** *Let R be complete intersection of codimension c and let M and N be finitely generated R -modules. If one has $\widehat{\mathrm{Tor}}_i^R(M, N) = 0$ for $c + 1$ consecutive values of i or $\mathrm{Tor}_i^R(M, N) = 0$ for $c + 1$ consecutive values of $i \geq 0$, then the following equality holds,*

$$\mathrm{CI}\text{-dim}_R(M \otimes_R^{\mathbf{L}} N) = \mathrm{CI}\text{-dim}_R M + \mathrm{CI}\text{-dim}_R N.$$

Proof. It follows from the theorem and Corollary (4.2) that the derived depth formula holds for M and N . For every R -complex X with $\mathrm{H}(X)$ finitely generated, there is an Auslander–Buchsbaum-type formula $\mathrm{CI}\text{-dim}_R X = \mathrm{depth} R - \mathrm{depth}_R X$; see [28, prop. 3.3]. Now the desired equality follows from the depth formula. \square

(4.9) **Theorem.** *Let R be of codepth c , let M be a finitely generated R -module of finite CI-dimension, and let N be an R -module. If one has $\widehat{\mathrm{Tor}}_i^R(M, N) = 0$ for $c + 1$ consecutive values of i or $\mathrm{Tor}_i^R(M, N) = 0$ for $c + 1$ consecutive values of $i > \mathrm{depth} R$, then the Tate homology modules $\widehat{\mathrm{Tor}}_i^R(M, N)$ vanish for all $i \in \mathbb{Z}$, and the derived depth formula holds for M and N .*

Proof. By [7, thm. (1.4) and (5.6)] the number $d = \mathrm{depth} R$ is an upper bound for the CI-dimension of M , and the complexity of M is at most c . If one has $\mathrm{Tor}_i^R(M, N) = 0$ for $c + 1$ consecutive values of $i > d$, then [25, cor. 2.3] yields $\mathrm{Tor}_i^R(M, N) = 0$ for all $i > d$, and then it follows from Lemma (4.3) and Theorem (2.3) that the derived depth formula holds for M and N .

Let $T \rightarrow P \rightarrow M$ be a complete resolution. By [7, lem. (1.5)] every syzygy of M has finite CI-dimension. In particular, $C_d(T)$ has finite CI-dimension, and it follows that $C_i(T)$ has finite CI-dimension for every $i \in \mathbb{Z}$. If one has $\widehat{\mathrm{Tor}}_i^R(M, N) = 0$ for $c + 1$ consecutive values of i , then there exists an $\iota \in \mathbb{Z}$ such that one has

$$0 = \mathrm{Tor}_{d+1}^R(C_\iota(T), N) = \mathrm{Tor}_{d+2}^R(C_\iota(T), N) = \cdots = \mathrm{Tor}_{d+c+1}^R(C_\iota(T), N).$$

Now [25, cor. 2.3] yields $\mathrm{Tor}_i^R(C_\iota(T), N) = 0$ for all $i > d$, and then it follows from Lemma (4.3) and Theorem (2.3) that the desired formula holds for M and N . \square

Proof of Lemma (4.3). The complex M has finite CI-dimension. If the homology complex $\mathrm{H}(M)$ is bounded, then it follows from [29, thm. 5.1.(b) and rmk. 2.5] that $\mathrm{Gpd}_R M$ is finite. Here we give a direct but similar argument that does not use boundedness of $\mathrm{H}(M)$.

Let $R \xrightarrow{\rho} R' \xleftarrow{\pi} Q$ be a quasi-deformation such that $\mathrm{pd}_Q(R' \otimes_R M)$ is finite. Assume, without loss of generality, that the rings Q and, therefore, R' are complete. Then Q has a dualizing complex D , and the complex $D' = \mathbf{R}\mathrm{Hom}_Q(R', D)$ is dualizing for R' ; see [24, sec. 3]. The complex $\mathrm{H}(D \otimes_Q^{\mathbf{L}} (R' \otimes_R M))$ is bounded above by [5, thm. 2.4.F and 2.3.F], and it follows from [24, prop. 7.3] that the complex M belongs to the Auslander category $\hat{\mathcal{A}}(R)$. Let x_1, \dots, x_c be a Q -regular sequence that generates $\mathrm{Ker} \pi$. The Koszul complex on x_1, \dots, x_c is a projective resolution of R' over Q , so one has $D' \simeq \Sigma^{-c}(D \otimes_Q^{\mathbf{L}} R')$. Now it is straightforward to verify that $\mathrm{H}(D' \otimes_{R'}^{\mathbf{L}} (R' \otimes_R M))$ is bounded above and that $R' \otimes_R M$ belongs to $\hat{\mathcal{A}}(R')$. Therefore, $g = \mathrm{Gpd}_{R'}(R' \otimes_R M)$ is finite by [24, thm. 8.1]. Let $P \xrightarrow{\simeq} M$ be a

semi-projective resolution over R . To prove that $\text{Gpd}_R M$ is finite, it suffices to show that the module $C_g(P)$ has finite Gorenstein projective dimension or, equivalently, finite Gorenstein flat dimension; cf. [11, thm. 4.1]. The quasi-isomorphism $R' \otimes_R P \rightarrow R' \otimes_R M$ is a semi-projective resolution over R' . By [30, thm. 3.4] the module $C_g(R' \otimes_R P) \cong R' \otimes_R C_g(P)$ is Gorenstein projective, in particular it is Gorenstein flat, and then so is $C_g(P)$ by [14, thm. A]. Thus $\text{Gpd}_R M$ is finite.

The implications

$$(iv) \Leftarrow (ii') \Leftarrow (i) \implies (ii) \implies (iii) \implies (iv)$$

are trivial or follow from (2.2.1). It remains to prove that (iv) implies (i).

Let $T \rightarrow P \rightarrow M$ be a complete resolution. The morphism $R' \otimes_R P \rightarrow R' \otimes_R M$ is then a semi-projective resolution over R' , and it is straightforward to verify that $R' \otimes_R T$ is a totally acyclic complex of projective R' -modules. Thus, by flatness of R' one has

$$R' \otimes_R \widehat{\text{Tor}}_i^R(M, N) \cong \widehat{\text{Tor}}_i^{R'}(R' \otimes_R M, R' \otimes_R N)$$

for all $i \in \mathbb{Z}$. Without loss of generality, assume that ρ is the identity map. The assumptions are now that R is isomorphic to $Q/(x_1, \dots, x_c)$, that $\text{pd}_Q M$ is finite, and that there is an integer h such that

$$(1) \quad 0 = \widehat{\text{Tor}}_h^R(M, N) = \widehat{\text{Tor}}_{h+1}^R(M, N) = \dots = \widehat{\text{Tor}}_{h+c}^R(M, N).$$

For $j \in \mathbb{Z}$ set $M_j = C_j(T)$. Notice that because R and M have finite projective dimension over Q , each module M_j has finite projective dimension over Q as well. Set $d = \text{depth } Q$; fix a j and let F be a projective resolution of M_j over Q of length $\text{pd}_Q M_j \leq d$. Let L be a semi-projective resolution of N over R . Without loss of generality, assume that one has $\text{sup } N = 0$. The filtrations \mathcal{F} and \mathcal{L} defined by

$$(\mathcal{F}^p(F \otimes_Q L))_n = \prod_{i \leq p} F_i \otimes_Q L_{n-i} \quad \text{and} \quad (\mathcal{L}^p(L \otimes_Q F))_n = \prod_{i \leq p} L_i \otimes_Q F_{n-i}$$

are bounded; they give rise to spectral sequences

$$\mathcal{F}E_{p,q}^2 \implies H_n(F \otimes_Q L) \quad \text{and} \quad \mathcal{L}E_{p,q}^2 \implies H_n(L \otimes_Q F).$$

The E^2 -terms are the iterated homologies of the underlying double complexes, obtained by first taking homology along columns and then along rows. The terms

$$\mathcal{F}E_{p,q}^2 = \text{Tor}_p^Q(M_j, H_q(N))$$

vanish for $q > 0$, and they vanish for p not in $\{0, \dots, d\}$. In particular, for $n > d$ one has $\mathcal{F}E_{i,n-i}^2 = 0$ for all $i \in \mathbb{Z}$. It follows that the homology modules $H_n(F \otimes_Q L)$ vanish for $n > d$; see [8, prop. 5.5]. From the isomorphism $H(F \otimes_R L) \cong H(L \otimes_R F)$ and [8, prop. 5.3a] one now gets

$$(2) \quad \mathcal{L}E_{i,n-i}^\infty = 0 \quad \text{for all } n > d \text{ and all } i \in \mathbb{Z}.$$

The homology in degree q within the p th column in the double complex $L \otimes_R F$ is isomorphic to $L_p \otimes_R H_q(R \otimes_Q^L M_j)$. The Koszul complex $K^Q(x_1, \dots, x_c)$ is a free resolution of R over Q , and the elements x_1, \dots, x_c act trivially on M_j , so one has $L_p \otimes_R H_q(M_j \otimes_Q^L R) \cong L_p \otimes_R M_j \binom{c}{q}$ for all q . Thus the E^2 -terms in the second sequence are

$$\mathcal{L}E_{p,q}^2 = \text{Tor}_p^R(N, M_j \binom{c}{q}) \cong \text{Tor}_p^R(M_j, N) \binom{c}{q}.$$

Clearly the terms $\mathcal{L}E_{p,q}^2$ vanish for q not in $\{0, \dots, c\}$, and $\mathcal{L}E_{p,q}^2$ vanishes for all q in $\{0, \dots, c\}$ if and only if it vanishes for one of them.

Assume now that $j < h - d + c - 1$ holds. One then has $h - j > 0$, and the hypothesis (1) yields $\mathcal{L}E_{p,q}^2 = 0$ for p in $\{h - j, \dots, h - j + c\}$. This yields the limit terms

$$(3) \quad \mathcal{L}E_{h-j+c+1,0}^2 = \mathcal{L}E_{h-j+c+1,0}^\infty \quad \text{and} \quad \mathcal{L}E_{h-j-1,c}^2 = \mathcal{L}E_{h-j-1,c}^\infty.$$

Moreover, the inequalities $h - j + c + 1 > d$ and $c + (h - j - 1) > d$ hold, so (2) and (3) combine to yield $\mathcal{L}E_{h-j+c+1,0}^2 = 0$ and $\mathcal{L}E_{h-j-1,c}^2 = 0$ and, therefore, $\mathcal{L}E_{h-j+c+1,q}^2 = 0 = \mathcal{L}E_{h-j-1,q}^2$ for all q .

Iterating this argument, one gets $\mathcal{L}E_{p,q}^2 = 0$ for all $p > 0$ and all q . Thus, for every $j < h - d + c - 1$ one has $0 = \text{Tor}_p^R(M_j, N) = \widehat{\text{Tor}}_{p+j}^R(M, N)$ for all $p > 0$; hence $\widehat{\text{Tor}}_i^R(M, N) = 0$ holds for all $i \in \mathbb{Z}$. \square

5. AUSLANDER'S DEPTH FORMULA

The purpose of this section is to connect the derived depth formula (1.4.1) with another generalization, due to Auslander [3], of the Auslander–Buchsbaum Formula.

In view of Lemma (4.3) the next result generalizes [10, thm. 3] by replacing finitely generated modules with complexes of modules with mild boundedness conditions and no assumptions of finite generation.

(5.1) Theorem. *Let M be an R -complex of finite Gorenstein projective dimension and let N be a bounded above R -complex. If one has $\widehat{\text{Tor}}_i^R(M, N) = 0$ for all $i \in \mathbb{Z}$, then $s = \sup \text{H}(M \otimes_R^{\mathbf{L}} N)$ is finite, there is an inequality,*

$$-s \leq \text{depth}_R M + \text{depth}_R N - \text{depth } R,$$

and equality holds if and only if $\text{depth}_R \text{H}_s(M \otimes_R^{\mathbf{L}} N)$ is zero. Moreover, if one has $s = 0$ or $\text{depth}_R \text{H}_s(M \otimes_R^{\mathbf{L}} N) \leq 1$, then there is an equality

$$\text{depth}_R \text{H}_s(M \otimes_R^{\mathbf{L}} N) - s = \text{depth}_R M + \text{depth}_R N - \text{depth } R.$$

Proof. By Theorem (2.3) the complex $\text{H}(M \otimes_R^{\mathbf{L}} N)$ is bounded above, and one has $\text{depth}_R(M \otimes_R^{\mathbf{L}} N) = \text{depth}_R M + \text{depth}_R N - \text{depth } R$; the inequality now follows from (1.3.1). By [17, 1.5.(3)] there is an isomorphism

$$\text{H}_{-s}(\mathbf{R}\text{Hom}_R(k, M \otimes_R^{\mathbf{L}} N)) \cong \text{Hom}_R(k, \text{H}_s(M \otimes_R^{\mathbf{L}} N)),$$

and it follows that the equality $\text{depth}_R(M \otimes_R^{\mathbf{L}} N) = -s$ holds if and only if the module $\text{H}_s(M \otimes_R^{\mathbf{L}} N)$ has depth zero. Finally, by [23, thm. 2.3] the desired equality $\text{depth}_R(M \otimes_R^{\mathbf{L}} N) = \text{depth}_R \text{H}_s(M \otimes_R^{\mathbf{L}} N) - s$ holds provided that one has $\text{depth}_R \text{H}_s(M \otimes_R^{\mathbf{L}} N) - s \leq \text{depth}_R \text{H}_i(M \otimes_R^{\mathbf{L}} N) - i$ for all $i \leq s$. \square

(5.2) The depth formula. Let M and N be finitely generated R -modules. Following Choi and Iyengar [10] we say that the *depth formula holds for M and N* if $s = \sup\{i \mid \text{Tor}_i^R(M, N) \neq 0\}$ is finite and one has

$$\text{depth}_R \text{Tor}_s^R(M, N) - s = \text{depth}_R M + \text{depth}_R N - \text{depth } R.$$

Auslander [3, thm. 1.2] proved that the formula holds, if M has finite projective dimension, and one has $s = 0$ or $\text{depth}_R \text{Tor}_s^R(M, N) \leq 1$.

(5.3) **Corollary.** *Let M and N be finitely generated R -modules, and assume that $s = 0$ or $\text{depth}_R \text{Tor}_s^R(M, N) \leq 1$. The depth formula holds for M and N if one of the following conditions is satisfied.*

- (a) M has finite G -dimension, and one has $\widehat{\text{Tor}}_i^R(M, N) = 0$ for all $i \in \mathbb{Z}$.
- (b) R is AB, and one has $\text{Tor}_i^R(M, N) = 0$ for $i \gg 0$.
- (c) R is complete intersection of codimension c , and one has $\widehat{\text{Tor}}_i^R(M, N) = 0$ for $c + 1$ consecutive values of i .
- (d) R is complete intersection of codimension c , and one has $\text{Tor}_i^R(M, N) = 0$ for $c + 1$ consecutive values of $i \geq 0$.
- (e) M has finite CI-dimension, and one has $\widehat{\text{Tor}}_i^R(M, N) = 0$ for codepth $R + 1$ consecutive values of $i \geq 0$.
- (f) M has finite CI-dimension, and one has $\text{Tor}_i^R(M, N) = 0$ for codepth $R + 1$ consecutive values of $i > \text{depth } R$.

Proof. Part (a) is immediate from Theorem (5.1). Part (b) follows from (a) in view of Proposition (3.2)(a). Similarly, part (c) follows in view of Theorem (4.6), part (d) follows in view of Corollary (4.2), and parts (e) and (f) follow in view of Theorem (4.9). \square

6. VANISHING OF COHOMOLOGY

Let M and N be finitely generated R -modules. If M has finite projective dimension or N has finite injective dimension, then the largest index for which $\text{Ext}_R^i(M, N)$ does not vanish is $i = \text{depth } R - \text{depth}_R M$. That is, the vanishing of cohomology $\text{Ext}_R^i(M, N)$ only depends on M ; see Ischebeck [22, 2.6]. The next result is more general, as Tate cohomology $\widehat{\text{Ext}}_R^*(M, N)$ vanishes if M has finite projective dimension or N has finite injective dimension, see (3.1), and finite projective/injective dimension implies finite G -dimension/Gorenstein injective dimension.

(6.1) **Theorem.** *Let M and N be finitely generated R -modules such that M has finite G -dimension or N has finite Gorenstein injective dimension. If one has $\widehat{\text{Ext}}_R^i(M, N) = 0$ for all $i \in \mathbb{Z}$, then the next equality holds*

$$\sup\{i \in \mathbb{Z} \mid \text{Ext}_R^i(M, N) \neq 0\} = \text{depth } R - \text{depth}_R M.$$

The proof is given at the end of the section; note that under the assumption that M has finite G -dimension, the desired equality follows from the proof of Theorem (3.6).

The notion of Gorenstein injective dimension is dual to that of Gorenstein projective dimension, and Tate cohomology $\widehat{\text{Ext}}_R^*(M, N)$ can be extended to the situation where the second variable N has finite Gorenstein injective dimension; see [2, 13].

Before we start the proof of Theorem (6.1) we record an easy consequence.

(6.2) **Remark.** Let M and N be finitely generated R -modules. Under each of the following conditions,

- (a) M has finite G -dimension and $\text{pd}_R N$ is finite
- (b) N has finite Gorenstein injective dimension and $\text{id}_R M$ is finite

Tate cohomology $\widehat{\text{Ext}}_R^*(M, N)$ vanishes by [30, thm. 4.5] and [2, thm. 3.9], whence one has $\sup\{i \in \mathbb{Z} \mid \text{Ext}_R^i(M, N) \neq 0\} = \text{depth } R - \text{depth}_R M$.

As a first step towards a proof of (6.1) we recall the notion of width.

(6.3) **Width.** The *width* of an R -complex M is defined as

$$\text{width}_R M = \inf \mathbf{H}(k \otimes_R^{\mathbf{L}} M).$$

There is an obvious inequality

$$(6.3.1) \quad \text{width}_R M \geq \inf \mathbf{H}(M),$$

and equality holds if $\mathbf{H}(M)$ is bounded below and degreewise finitely generated.

(6.4) **Proposition.** *Let N be an R -complex of finite Gorenstein injective dimension and let M be a bounded above R -complex. If one has $\widehat{\text{Ext}}_R^i(M, N) = 0$ for all $i \in \mathbb{Z}$, then the next equality holds,*

$$\text{width}_R \mathbf{RHom}_R(M, N) = \text{depth}_R M + \text{width}_R N - \text{depth } R.$$

By [13, (5.6.1)] the homology complex $\mathbf{H}(\mathbf{RHom}_R(M, N))$ is bounded below.

Proof. Choose a complete injective resolution $N \rightarrow I \xrightarrow{v} U$ and let $\pi: P \xrightarrow{\simeq} M$ be a semi-projective resolution. The induced quasi-isomorphism

$$\text{Hom}_R(\pi, I): \text{Hom}_R(M, I) \xrightarrow{\simeq} \text{Hom}_R(P, I)$$

is a semi-injective resolution, and the Künneth formula yields

$$\begin{aligned} \mathbf{H}(\mathbf{RHom}_R(k, \mathbf{RHom}_R(M, N))) &\cong \mathbf{H}(\text{Hom}_R(k, \text{Hom}_R(P, I))) \\ &\cong \mathbf{H}(\text{Hom}_R(k \otimes_R P, I)) \\ &\cong \mathbf{H}(\text{Hom}_k(k \otimes_R P, \text{Hom}_R(k, I))) \\ &\cong \text{Hom}_k(\mathbf{H}(k \otimes_R^{\mathbf{L}} M), \mathbf{H}(\mathbf{RHom}_R(k, N))). \end{aligned}$$

It follows from (6.3.1) and Proposition (1.5) that $\text{width}_R \mathbf{RHom}_R(M, N)$ is finite if and only if $\text{width}_R M$ and $\text{depth}_R N$ are both finite. In particular, the left- and right-hand sides of the desired equality are simultaneously finite.

Assume that $\text{width}_R M$ and $\text{depth}_R N$ are finite. Set $K = \Sigma^{-1}(\text{Cone } \iota)$; consider the degreewise split exact sequence $0 \rightarrow \Sigma^{-1}U \rightarrow K \rightarrow I \rightarrow 0$ and apply the functor $\text{Hom}_R(M, -)$. By assumption, the complex $\text{Hom}_R(M, U)$ is acyclic, cf. [13, def. (5.5)], so there is a quasi isomorphism $\text{Hom}_R(M, K) \xrightarrow{\simeq} \text{Hom}_R(M, I)$. In low degrees, K is isomorphic to the mapping cone of an isomorphism. Therefore, there exist homomorphisms $\sigma_i: K_i \rightarrow K_{i+1}$ such that $1^{K_i} = \sigma_{i-1} \partial_i^K + \partial_{i+1}^K \sigma_i$ holds for $i \ll 0$. Since K is a complex of injective modules, and one has $\partial_{i+1}^K \sigma_i = 1^{\text{Im } \partial_{i+1}^K}$ for $i \ll 0$, it follows that the modules $\text{Ker } \partial_i^K = \text{Im } \partial_{i+1}^K \cong C_{i+1}(K)$ are injective for $i \ll 0$. Fix $n \ll 0$; the subcomplex $J = \cdots \rightarrow K_{n+2} \rightarrow K_{n+1} \rightarrow \text{Im } \partial_{n+1}^K \rightarrow 0$ consists of injective modules, the sequence $0 \rightarrow J \rightarrow K \rightarrow K/J \rightarrow 0$ is split exact, and the quotient complex $K/J = 0 \rightarrow C_n(K) \rightarrow K_{n-1} \rightarrow \cdots$ is contractible. It follows that there are quasi-isomorphisms,

$$(1) \quad \text{Hom}_R(M, J) \xrightarrow{\simeq} \text{Hom}_R(M, K) \xrightarrow{\simeq} \text{Hom}_R(M, I).$$

Choose a semi-injective resolution $\iota': M \xrightarrow{\simeq} I'$, where ι' is injective and I' is bounded above; see (1.2)(I). Then the complex $C = \text{Coker } \iota'$ is bounded above and acyclic. Apply $\text{Hom}_R(-, J)$ to the exact sequence $0 \rightarrow M \xrightarrow{\iota'} I' \rightarrow C \rightarrow 0$ of

R -complexes. The complex $\mathrm{Hom}_R(C, J)$ is acyclic by [11, lem. 2.5], so there is a quasi-isomorphism

$$(2) \quad \mathrm{Hom}_R(I', J) \xrightarrow{\simeq} \mathrm{Hom}_R(M, J).$$

The complex $\mathrm{Hom}_R(I', J)$ is bounded below and consists of flat R -modules, so it is semi-flat in the sense of [5, 6], and it follows from (1) and (2) that there is a quasi-isomorphism $\mathrm{Hom}_R(I', J) \xrightarrow{\simeq} \mathrm{Hom}_R(M, J)$. The third equality in the next chain uses homomorphism evaluation, a variation on [5, lem. 4.4.I] which is proved similarly to Lemma (1.1).

$$\begin{aligned} \mathrm{width}_R \mathbf{R}\mathrm{Hom}_R(M, N) &= \inf \mathrm{H}(k \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_R(M, N)) \\ &= \inf \mathrm{H}(k \otimes_R \mathrm{Hom}_R(I', J)) \\ &= \inf \mathrm{H}(\mathrm{Hom}_R(\mathrm{Hom}_R(k, I'), J)) \\ &= \inf \mathrm{H}(\mathrm{Hom}_k(\mathrm{Hom}_R(k, I'), \mathrm{Hom}_R(k, J))) \\ &= \inf \mathrm{H}(\mathrm{Hom}_R(k, J)) - \sup \mathrm{H}(\mathrm{Hom}_R(k, I')) \\ &= \inf \mathrm{H}(\mathrm{Hom}_R(k, J)) + \mathrm{depth}_R M. \end{aligned}$$

For $M = R$ this equality reads $\mathrm{width}_R N = \inf \mathrm{H}(\mathrm{Hom}_R(k, J)) + \mathrm{depth}_R R$, and the desired equality follows. \square

A dual argument yields the next result, which is also invoked in the proof of Theorem (6.1). The special case of Proposition (6.5) where M and N are finitely generated, which is the context of (6.1), follows from the proof of Theorem (3.6).

(6.5) Proposition. *Let M be an R -complex of finite Gorenstein projective dimension and let N be a bounded below R -complex. If one has $\widehat{\mathrm{Ext}}_R^i(M, N) = 0$ for all $i \in \mathbb{Z}$, then the next equality holds,*

$$\mathrm{width}_R \mathbf{R}\mathrm{Hom}_R(M, N) = \mathrm{depth}_R M + \mathrm{width}_R N - \mathrm{depth}_R R. \quad \square$$

Note that by (3.1.1) the homology complex $\mathrm{H}(\mathbf{R}\mathrm{Hom}_R(M, N))$ is bounded below.

Proof of Theorem (6.1). It follows from [13, (5.6.1)] and (3.1.1) that the complex $\mathbf{R}\mathrm{Hom}_R(M, N)$ has bounded below homology. Moreover, each homology module $\mathrm{H}_{-i}(\mathbf{R}\mathrm{Hom}_R(M, N)) = \mathrm{Ext}_R^i(M, N)$ is finitely generated, as M and N are finitely generated. Now (6.3.1) and Propositions (6.4) and (6.5) yield

$$\begin{aligned} \sup\{i \in \mathbb{Z} \mid \mathrm{Ext}_R^i(M, N) \neq 0\} &= -\inf \mathbf{R}\mathrm{Hom}_R(M, N) \\ &= -\mathrm{width}_R \mathbf{R}\mathrm{Hom}_R(M, N) \\ &= -\mathrm{depth}_R M - \mathrm{width}_R N + \mathrm{depth}_R R \\ &= \mathrm{depth}_R R - \mathrm{depth}_R M. \quad \square \end{aligned}$$

ACKNOWLEDGMENTS

We thank Olgur Celikbas and Srikanth Iyengar for their comments on an earlier version of this paper. We also acknowledge an anonymous referee whose pertinent suggestions helped us tighten the presentation.

REFERENCES

1. Tokuji Araya and Yuji Yoshino, *Remarks on a depth formula, a grade inequality and a conjecture of Auslander*, *Comm. Algebra* **26** (1998), no. 11, 3793–3806. MR1647079
2. Javad Asadollahi and Shokrollah Salarian, *Cohomology theories based on Gorenstein injective modules*, *Trans. Amer. Math. Soc.* **358** (2006), no. 5, 2183–2203. MR2197453
3. M. Auslander, *Modules over unramified regular local rings*, *Illinois J. Math.* **5** (1961), 631–647. MR0179211
4. Luchezar L. Avramov and Ragnar-Olaf Buchweitz, *Support varieties and cohomology over complete intersections*, *Invent. Math.* **142** (2000), no. 2, 285–318. MR1794064
5. Luchezar L. Avramov and Hans-Bjørn Foxby, *Homological dimensions of unbounded complexes*, *J. Pure Appl. Algebra* **71** (1991), no. 2-3, 129–155. MR1117631
6. Luchezar L. Avramov, Hans-Bjørn Foxby, and Stephen Halperin, *Differential graded homological algebra*, preprint (1994–2011).
7. Luchezar L. Avramov, Vesselin N. Gasharov, and Irena V. Peeva, *Complete intersection dimension*, *Inst. Hautes Études Sci. Publ. Math.* (1997), no. 86, 67–114 (1998). MR1608565
8. Henri Cartan and Samuel Eilenberg, *Homological algebra*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1999, With an appendix by David A. Buchsbaum, Reprint of the 1956 original. MR1731415
9. Olgur Celikbas and Hailong Dao, *Necessary conditions for the depth formula over Cohen–Macaulay local rings*, *J. Pure Appl. Algebra* **218** (2014), no. 3, 522–530. MR3124216
10. Sangki Choi and Srikanth Iyengar, *On a depth formula for modules over local rings*, *Comm. Algebra* **29** (2001), no. 7, 3135–3143. MR1849128
11. Lars Winther Christensen, Anders Frankild, and Henrik Holm, *On Gorenstein projective, injective and flat dimensions—A functorial description with applications*, *J. Algebra* **302** (2006), no. 1, 231–279. MR2236602
12. Lars Winther Christensen and Henrik Holm, *Ascent properties of Auslander categories*, *Canad. J. Math.* **61** (2009), no. 1, 76–108. MR2488450
13. Lars Winther Christensen and David A. Jorgensen, *Tate (co)homology via pinched complexes*, *Trans. Amer. Math. Soc.* **366** (2014), no. 2, 667–689. MR3130313
14. Lars Winther Christensen and Sean Sather-Wagstaff, *Transfer of Gorenstein dimensions along ring homomorphisms*, *J. Pure Appl. Algebra* **214** (2010), no. 6, 982–989. MR2580673
15. W. G. Dwyer and J. P. C. Greenlees, *Complete modules and torsion modules*, *Amer. J. Math.* **124** (2002), no. 1, 199–220. MR1879003
16. Hans-Bjørn Foxby, *Homological dimensions of complexes of modules*, Séminaire d’Algèbre Paul Dubreil et Marie-Paule Malliavin, 32ème année (Paris, 1979), Springer, Berlin, 1980, Lecture Notes in Mathematics, no. 795, pp. 360–368.
17. Hans-Bjørn Foxby and Srikanth Iyengar, *Depth and amplitude for unbounded complexes*, *Commutative algebra (Grenoble/Lyon, 2001)* (Providence, RI), *Contemp. Math.*, vol. 331, Amer. Math. Soc., 2003, pp. 119–137. MR2013162
18. Craig Huneke and David A. Jorgensen, *Symmetry in the vanishing of Ext over Gorenstein rings*, *Math. Scand.* **93** (2003), no. 2, 161–184. MR2009580
19. Craig Huneke, David A. Jorgensen, and Roger Wiegand, *Vanishing theorems for complete intersections*, *J. Algebra* **238** (2001), no. 2, 684–702. MR1823780
20. Craig Huneke and Roger Wiegand, *Tensor products of modules and the rigidity of Tor*, *Math. Ann.* **299** (1994), no. 3, 449–476. MR1282227
21. Alina Iacob, *Absolute, Gorenstein, and Tate torsion modules*, *Comm. Algebra* **35** (2007), no. 5, 1589–1606. MR2317632
22. Friedrich Ischebeck, *Eine Dualität zwischen den Funktoren Ext und Tor*, *J. Algebra* **11** (1969), 510–531. MR0237613
23. Srikanth Iyengar, *Depth for complexes, and intersection theorems*, *Math. Z.* **230** (1999), no. 3, 545–567. MR1680036
24. Srikanth Iyengar and Henning Krause, *Acyclicity versus total acyclicity for complexes over Noetherian rings*, *Doc. Math.* **11** (2006), 207–240. MR2262932
25. David A. Jorgensen, *Vanishing of (co)homology over commutative rings*, *Comm. Algebra* **29** (2001), no. 5, 1883–1898. MR1837948
26. David A. Jorgensen and Liana M. Şega, *Nonvanishing cohomology and classes of Gorenstein rings*, *Adv. Math.* **188** (2004), no. 2, 470–490. MR2087235

27. Parviz Sahandi, Tirdad Sharif, and Siamak Yassemi, *Depth formula via complete intersection flat dimension*, *Comm. Algebra* **39** (2011), no. 11, 4002–4013. MR2855108
28. Sean Sather-Wagstaff, *Complete intersection dimensions for complexes*, *J. Pure Appl. Algebra* **190** (2004), no. 1-3, 267–290. MR2043332
29. Sean Sather-Wagstaff, *Complete intersection dimensions and Foxby classes*, *J. Pure Appl. Algebra* **212** (2008), no. 12, 2594–2611. MR2452313
30. Oana Veliche, *Gorenstein projective dimension for complexes*, *Trans. Amer. Math. Soc.* **358** (2006), no. 3, 1257–1283. MR2187653

DEPARTMENT OF MATH. AND STAT., TEXAS TECH UNIVERSITY, LUBBOCK, TX 79409, U.S.A.

E-mail address: `lars.w.christensen@ttu.edu`

URL: `http://www.math.ttu.edu/~lchrste`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS, ARLINGTON, TX 76019, U.S.A.

E-mail address: `djorgens@uta.edu`

URL: `http://dreadnought.uta.edu/~dave`