# THE GOLOD PROPERTY OF POWERS OF THE MAXIMAL IDEAL OF A LOCAL RING

LARS WINTHER CHRISTENSEN AND OANA VELICHE

Abstract. We identify minimal cases in which a power $\mathfrak{m}^i \neq 0$ of the maximal ideal of a local ring $R$ is not Golod, i.e. the quotient ring $R/\mathfrak{m}^i$ is not Golod. Complementary to a 2014 result by Rossi and Şega, we prove that for a generic artinian Gorenstein local ring with $\mathfrak{m}^4 = 0 \neq \mathfrak{m}^3$, the quotient $R/\mathfrak{m}^3$ is not Golod. This is provided that $\mathfrak{m}$ is minimally generated by at least 3 elements. Indeed, we show that if $\mathfrak{m}$ is 2-generated, then every power $\mathfrak{m}^i \neq 0$ is Golod.

## 1. Introduction

In this paper a local ring is a commutative noetherian ring $R$ with unique maximal ideal $\mathfrak{m}$. Such a ring is called *Golod* if the ranks of the modules in the minimal free resolution of the residue field $R/\mathfrak{m}$ attain the upper bound established by Serre; the precise definition is recalled in Section 3.

The field $R/\mathfrak{m}$ is trivially Golod, and so is the quotient ring $R/\mathfrak{m}^2$; see for example Avramov's exposition [3, prop. 5.2.4]. Moreover, if $R$ is a regular local ring, then the quotient $R/\mathfrak{m}^i$ is Golod for every $i \geqslant 1$. Rossi and Şega [20] prove that for a generic artinian Gorenstein local ring $(R, \mathfrak{m})$ with $\mathfrak{m}^4 \neq 0$, every proper quotient $R/\mathfrak{m}^i$ is Golod.

In this note we provide minimal examples of local rings with proper quotients $R/\mathfrak{m}^i$ that are *not* Golod. They come out of an investigation of the complementary case to above mentioned result from [20]. The following extract from Theorem (4.2) points to a whole family of local rings with $\mathfrak{m}^4 = 0$ and $R/\mathfrak{m}^3$ not Golod. In fact, this is the behavior of generic graded Gorenstein local $k$-algebras of socle degree 3.

(1.1) **Theorem.** *Let $k$ be a field; set $Q = k[\![x, y, z]\!]$ and $\mathfrak{n} = (x, y, z)$. Let $I \subseteq \mathfrak{n}^2$ be a homogeneous Gorenstein ideal in $Q$ with $\mathfrak{n}^4 \subseteq I \not\supseteq \mathfrak{n}^3$ and set $(R, \mathfrak{m}) = (Q/I, \mathfrak{n}/I)$. The following conditions are equivalent.*

- $(i)$ *$I$ is generated by quadratic forms.*
- $(ii)$ *$R$ is Koszul, i.e. the minimal free resolution of $k$ over $R$ is linear.*
- $(iii)$ *$R$ is complete intersection.*
- $(iv)$ *$R$ has an exact zero divisor, i.e. an element $a \neq 0$ with $(0 : a)$ principal.*
- $(v)$ *$R/\mathfrak{m}^3$ is not Golod.*

*Date*: 16 December 2017.

2010 *Mathematics Subject Classification.* Primary 13H10. Secondary 13D02.

*Key words and phrases.* Artinian Gorenstein ring, exact zero-divisor, Golod ring, Koszul ring.

L.W.C. was partly supported by NSA grant H98230-14-0140 and Simons Foundation collaboration grant 428308.

To discuss in which sense these rings are generic and constitute minimal examples of local rings $(R, \mathfrak{m})$ with proper quotients $R/\mathfrak{m}^i$ that are not Golod, we start to introduce the terminology that will be used throughout the paper.

* * *

Let $(R, \mathfrak{m})$ be a local ring with the residue field $k = R/\mathfrak{m}$.

(1.2) As remarked above, the quotient rings $R/\mathfrak{m}$ and $R/\mathfrak{m}^2$ are both Golod.
Thus, for a proper quotient $R/\mathfrak{m}^i$ to be not Golod, one must have $i \geqslant 3$.

(1.3) For an ideal $I \subseteq R$ we denote by $\mu(I)$ its minimal number of generators. The number $\mu(\mathfrak{m})$ is the *embedding dimension* of $R$. It is known that every local ring of embedding dimension 1 is Golod; see (1.4). Further, we prove in Theorem (2.2) that if $R$ has embedding dimension 2, then every proper quotient $R/\mathfrak{m}^i$ is Golod.

Thus, for a proper quotient $R/\mathfrak{m}^i$ to be not Golod, the embedding dimension of $R$ must be at least 3.

(1.4) The embedding dimension $e$ and the Krull dimension $d$ of $R$ satisfy $e \geqslant d$, with equality if and only if $R$ is regular. The difference $e - d$ is called the *codimension* of $R$. If $R$ has codimension at most 1, then $R$ is Golod, see [3, prop. 5.2.5], and $R/\mathfrak{m}^i$ is known to be Golod for every $i \geqslant 1$ by work of Şega [22, prop. 6.10].

Thus, for a proper quotient $R/\mathfrak{m}^i$ to be not Golod, the codimension of $R$ must be at least 2.

We exhibit in Example (2.4) a complete intersection local ring $(R, \mathfrak{m})$ of embedding dimension 3 and codimension 2 with $R/\mathfrak{m}^i$ not Golod for all $i \geqslant 3$. Thus, among local rings $(R, \mathfrak{m})$ with the property that a proper quotient $R/\mathfrak{m}^i$ is not Golod, this ring is minimal with regard to codimension. It has dimension 1; a 0-dimensional, i.e. artinian, local ring with the property must have codimension at least 3.

(1.5) Let $(R, \mathfrak{m})$ be artinian. The integer $s$ with $\mathfrak{m}^{s+1} = 0 \neq \mathfrak{m}^s$ is called the *socle degree* of $R$. The *type* of $R$ is the dimension of the socle as a vector space over the residue field; i.e. type $R = \dim_k(0 : \mathfrak{m})$.

In view of (1.2), the ring $R$ must have socle degree at least 3 for a proper quotient $R/\mathfrak{m}^i$ to be not Golod.

It follows that among local rings $(R, \mathfrak{m})$ with a proper quotient $R/\mathfrak{m}^i$ that is not Golod, the rings in Theorem (1.1) are minimal with regard to dimension and embedding dimension, and then with regard to socle degree and type.

(1.6) An element $a \neq 0$ in a commutative ring $R$ is called an *exact zero divisor* if the annihilator $(0 : a)$ is a principal ideal. Exact zero divisors in artinian Gorenstein local rings of socle degree 3 were studied in depth by Henriques and Şega in [6].

A generic artinian Gorenstein local graded $k$-algebra of socle degree 3 has an exact zero divisor; this follows from work of Conca, Rossi, and Valla [12]; see [6, rmk. 4.3][1]. In particular, generic Gorenstein algebras of embedding dimension 3 and socle degree 3 satisfy the conditions in Theorem (1.1). We provide examples of Gorenstein algebras of socle degree 3 without exact zero divisors in (3.4) and (4.3).

[1]The notion of *generic* that is used in [12] and [6] is, at least formally, different from the one used in [20]. However, Theorem (4.2) and thereby Theorem (1.1) apply to rings that are generic in either sense; see also the discussion after (3.2).

## 2. Embedding dimension 2

We prove that all proper quotients $R/\mathfrak{m}^i$ are Golod for any local ring $(R, \mathfrak{m})$ of embedding dimension 2. To frame this result we exhibit a local ring of embedding dimension 3 and codimension 2 such that $R/\mathfrak{m}^i$ is not Golod for $i \geqslant 3$.

(2.1) Let $(R, \mathfrak{m})$ be a local ring. The *valuation* of an ideal $J \subset R$ is the largest integer $i$ with $J \subseteq \mathfrak{m}^i$; it is written $v_R(J)$. For an element $x \in R$ one sets $v_R(x) = v_R((x))$.

In [21] Scheja shows that a local ring of embedding dimension 2 is either complete intersection or Golod; see also [3, prop. (5.3.4)]. The gist of the next result is that such a complete intersection cannot arise as a proper quotient by a power of the maximal ideal.

(2.2) **Theorem.** *Let $(R, \mathfrak{m})$ be a local ring of embedding dimension 2. For every $i \geqslant 1$ with $\mathfrak{m}^i \neq 0$, the quotient ring $R/\mathfrak{m}^i$ is Golod.*

**Proof.** Let $(\widehat{R}, \widehat{\mathfrak{m}})$ be the $\mathfrak{m}$-adic completion of $R$. One has $R/\mathfrak{m}^i \cong \widehat{R}/\widehat{\mathfrak{m}}^i$, so we may assume that $R$ is complete. By Cohen's Structure Theorem there is a regular local ring $(Q, \mathfrak{n})$ of embedding dimension 2 and an ideal $J \subseteq \mathfrak{n}^2$ with $R \cong Q/J$. Thus, one has $R/\mathfrak{m}^i \cong Q/(J + \mathfrak{n}^i)$, and $\mathfrak{m}^i \neq 0$ if and only if $\mathfrak{n}^i \nsubseteq J$. Per (1.2) we may assume that one has $i \geqslant 3$. To prove that $R/\mathfrak{m}^i$ is Golod, it suffices to argue that it is not complete intersection. Thus, we now argue that if $\mathfrak{n}^i \nsubseteq J$, then $J + \mathfrak{n}^i$ cannot be generated by two elements.

For every $i$ the ideal $\mathfrak{n}^i$ is minimally generated by $i + 1$ elements. In particular, if $J \subseteq \mathfrak{n}^i$, then $J + \mathfrak{n}^i$ cannot be generated by 2 elements. We now assume that $J$ is not contained in $\mathfrak{n}^i$; setting $t = v_Q(J)$ this means $t + 1 \leqslant i$. Write

$$J = (f_1, \ldots, f_n)$$

where $v_Q(f_1) = t$, and assume towards a contradiction that $J + \mathfrak{n}^i$ is generated by 2 elements. As $\mathfrak{n}(J + \mathfrak{n}^i) \subseteq \mathfrak{n}^{t+1}$ and $f_1 \in \mathfrak{n}^t \setminus \mathfrak{n}^{t+1}$ we may assume that $J + \mathfrak{n}^i$ is minimally generated by $f_1$ and some element $g$. One has $f_n = af_1 + bg$ for $a, b \in Q$, and $b$ is not a unit as $g \notin J$. Now write

$$g = \sum_{j=1}^{n} a_j f_j + h$$

with $h \in \mathfrak{n}^i$; without loss of generality we may assume $a_1 = 0$. Now one has

$$f_n = af_1 + b\left(\sum_{j=2}^{n} a_j f_j + h\right),$$

whence $f_n(1 - a_n b)$ belongs to $(f_1, \ldots, f_{n-1}) + \mathfrak{n}^i$. As $1 - a_n b$ is a unit, this yields

$$J + \mathfrak{n}^i = (f_1, \ldots, f_{n-1}) + \mathfrak{n}^i .$$

By recursion, one gets $J + \mathfrak{n}^i = (f_1) + \mathfrak{n}^i$, and it follows from Lemma (2.3) that $J + \mathfrak{n}^i$ cannot be generated by 2 elements; a contradiction. $\square$

(2.3) **Lemma.** *Let $(Q, \mathfrak{n})$ be a regular local ring of embedding dimension 2. If $f \in \mathfrak{n}^2$ and $i \geqslant 2$ are such that $\mathfrak{n}^i \nsubseteq (f)$, then one has $\mu((f) + \mathfrak{n}^i) > 2$.*

**Proof.** If $f \in \mathfrak{n}^i$, then the statement is clear as one has $\mu(\mathfrak{n}^i) = i + 1 \geqslant 3$; thus we may assume that $f \notin \mathfrak{n}^i$. Assume towards a contradiction that $(f) + \mathfrak{n}^i$ is minimally generated by two elements. Set $t = v_Q(f)$; as one has $\mathfrak{n}((f) + \mathfrak{n}^i) \subseteq \mathfrak{n}^{t+1}$ and $f \in \mathfrak{n}^t \setminus \mathfrak{n}^{t+1}$ we may assume that $(f) + \mathfrak{n}^i$ is minimally generated by $f$ and some element $g$ of valuation $i$. Let $x$ and $y$ be minimal generators of $\mathfrak{n}$ and write

$$g = \sum_{j=0}^{i} a_j x^{i-j} y^j \qquad \text{and} \qquad x^{i-j}y^j = b_j f + c_j g$$

with $a_j, b_j, c_j \in Q$. These expressions yield $g = \big(\sum_{j=0}^{i} a_j b_j\big) f + \big(\sum_{j=0}^{i} a_j c_j\big) g$, so if $c_j \in \mathfrak{n}$ for all $j$ one has $g = \big(1 - \sum_{j=0}^{i} a_j c_j\big)^{-1} \big(\sum_{j=0}^{i} a_j b_j\big) f$, which contradicts the assumption that $f$ and $g$ minimally generate $(f) + \mathfrak{n}^i$. Thus, $c_j$ is a unit for some $j$, whence $(f)+\mathfrak{n}^i$ is minimally generated by $f$ and $x^{i-j}y^j$. By symmetry in $x$ and $y$ we may assume that $j \geqslant 1$. Write $x^{i-j+1}y^{j-1} = af + bx^{i-j}y^j$ with $a, b \in Q$; one then has

$$af = x^{i-j}y^{j-1}(x - by) \, .$$

First consider the case $j = i$ and recall that $i \geqslant 2$. Notice that $f \in (y)$ would force $(f) + \mathfrak{n}^i = (f, y^i) \subseteq (y)$, which is absurd, so $f$ is not divisible by $y$. Since $Q$ is a domain and $(y)$ is a prime ideal, it follows that $a$ is divisible by $y^{j-1}$. Writing $a = a'y^{j-1}$ one now has $a'f = x - by$, which is absurd as $f \in \mathfrak{n}^2$ and $x, y$ minimally generate $\mathfrak{n}$.

Finally consider the case $j < i$. Notice as above that $f$ is not divisible by $x$ nor by $y$. Therefore, one has $a = a'x^{i-j}y^{j-1}$ and $a'f = x - by$, which again contradicts the assumption that $x$ and $y$ minimally generate $\mathfrak{n}$. □

Apart from showing that quotients $R/\mathfrak{m}^i$ of a codimension 2 local ring may not be Golod, the next example shows that there are artinian local rings of embedding dimension 3 and any socle degree $s \geqslant 3$ with $R/\mathfrak{m}^i$ not Golod for $3 \leqslant i \leqslant s$.

(2.4) **Example.** Let $k$ be a field and consider the codimension 2 complete intersection local ring $R = k[\![x, y, z]\!]/(x^2, y^2)$ with maximal ideal $\mathfrak{m} = (x, y, z)/(x^2, y^2)$. For $i \geqslant 3$ it is elementary to verify that the quotient ring

$$R/\mathfrak{m}^i = \frac{k[\![x, y, z]\!]}{(x^2, y^2, z^i, xz^{i-1}, yz^{i-1}, xyz^{i-2})}$$

is not Golod. Indeed, the Koszul complex $K$ on the generators $x, y, z$ of $\mathfrak{m}$ is the exterior algebra of the free module with basis $\varepsilon_x, \varepsilon_y, \varepsilon_z$. The element $xy(\varepsilon_x \wedge \varepsilon_y)$ in $K_2$ is the product of the cycles $x\varepsilon_x, y\varepsilon_y \in K_1$, and it is not a boundary as one has

$$\partial(w(\varepsilon_x \wedge \varepsilon_y \wedge \varepsilon_z)) = xw(\varepsilon_y \wedge \varepsilon_z) - yw(\varepsilon_x \wedge \varepsilon_z) + zw(\varepsilon_x \wedge \varepsilon_y) \, ,$$

and $xy \notin (z)$. Thus there is a non-trivial product in Koszul homology, which by Golod's original work [13] means that $R/\mathfrak{m}^i$ is not Golod.

Notice that for $s \geqslant 3$ the artinian local ring $(\dot{R}, \dot{\mathfrak{m}}) = (R/\mathfrak{m}^{s+1}, \mathfrak{m}/\mathfrak{m}^{s+1})$ has $\dot{R}/\dot{\mathfrak{m}}^i = R/\mathfrak{m}^i$ not Golod for $3 \leqslant i \leqslant s$.

## 3. Artinian Gorenstein rings

Let $(R, \mathfrak{m}, k)$ be a local ring, i.e. $k$ is the residue field $R/\mathfrak{m}$. For a finitely generated $R$-module $M$, the power series $\mathrm{P}_M^R(z) = \sum_{j=0}^{\infty}(\dim_k \mathrm{Tor}_j^R(k, M))z^j$ is called the *Poincaré series* of $M$ over $R$. The numbers $\dim_k \mathrm{Tor}_*^R(k, M)$ are known as the

*Betti numbers* of $M$; they record the ranks of the free modules in the minimal free resolution of $M$ over $R$. In particular, $R$ is Golod if and only if one has

$$\mathrm{P}_k^R(z) = \frac{(1+z)^e}{1 - \sum_{j=1}^{e-d} (\dim_k \mathrm{H}_j(\mathrm{K}^R)) z^{j+1}} \tag{3.0.1}$$

where $e$ and $d$ are the embedding dimension and depth of $R$, and $\mathrm{K}^R$ is the Koszul complex on a minimal set of generators of $\mathfrak{m}$; see [3, (5.0.1)].

It is standard to refer to $\mathrm{P}_k^R(z)$ as the Poincaré series of $R$. For an artinian Gorenstein local ring of embedding dimension $e \geqslant 2$ and socle degree $s$, the Poincaré series of $R/\mathfrak{m}^s$ was computed by Avramov and Levin [19, thm. 2]:

$$\mathrm{P}_k^{R/\mathfrak{m}^s}(z) = \frac{\mathrm{P}_k^R(z)}{1 - z^2\,\mathrm{P}_k^R(z)} \; . \tag{3.0.2}$$

The special case of a complete intersection was first done by Gulliksen [15, thm. 1]:

$$\mathrm{P}_k^{R/\mathfrak{m}^s}(z) = \frac{1}{(1-z)^e - z^2} \; . \tag{3.0.3}$$

(3.1) **Proposition.** *If $(R,\mathfrak{m})$ is an artinian complete intersection local ring of embedding dimension at least 3 and socle degree $s$, then $R/\mathfrak{m}^s$ is not a Golod ring.*

**Proof.** Let $e$ be the embedding dimension of $R$. By (3.0.3) one has

$$\mathrm{P}_k^{R/\mathfrak{m}^s}(z) = \frac{1}{(1-z)^e - z^2} = \frac{(1+z)^e}{(1-z^2)^e - z^2(1+z)^e} \; ;$$

notice that the denominator has degree $2e$ since $e \geqslant 3$. If $R/\mathfrak{m}^s$ were Golod, then by (3.0.1) the Poincaré series would have the form $(1+z)^e/d(z)$ where the denominator $d(z)$ has degree $e+1 < 2e$. Thus, $R/\mathfrak{m}^s$ is not a Golod ring. □

(3.2) Let $(R,\mathfrak{m},k)$ be artinian of embedding dimension $e$ and socle degree $s$.

By Cohen's Structure Theorem there is a regular local ring $(Q,\mathfrak{n})$ and an ideal $I$ with $\mathfrak{n}^{s+1} \subseteq I \subseteq \mathfrak{n}^2$ such that $Q/I \cong R$. This is called the *minimal Cohen presentation* of $R$; notice in particular that $Q$ also has embedding dimension $e$.

Denote by $h_R(i)$ and $\mathrm{H}_R(z)$ the Hilbert function and Hilbert series of $R$; i.e.

$$h_R(i) = \dim_k(\mathfrak{m}^i/\mathfrak{m}^{i+1}) \text{ for } i \geqslant 0 \qquad \text{and} \qquad \mathrm{H}_R(z) = \sum_{i=0}^{s} h_R(i) z^i \; .$$

One says that $R$ is *Koszul* if the associated graded $k$-algebra $\bigoplus_{i=0}^s \mathfrak{m}^i/\mathfrak{m}^{i+1}$ is Koszul in the traditional sense that $k$ has a linear resolution; see the discussion in [6, 1.10]. If $R$ is Koszul, then one has $\mathrm{P}_k^R(z)\,\mathrm{H}_R(-z) = 1$.

If $R$ is Gorenstein, then for every $i \geqslant 0$ there is an inequality

$$h_R(i) \;\leqslant\; \min\{h_Q(i), h_Q(s-i)\} \;=\; \min\left\{ \binom{e-1+i}{e-1}, \binom{e-1+s-i}{e-1} \right\} \; ;$$

If equality holds for every $i$, then $R$ is called *compressed*, see [20, sec. 4].

The idea of compressed rings was introduced by Iarrobino, in [16, thm. I] he shows that generic artinian Gorenstein local standard graded algebras over a field are compressed. Rossi and Şega prove [20, prop. 6.3] that for a compressed artinian Gorenstein local ring $(R,\mathfrak{m})$ of socle degree $s \neq 3$ the quotient ring $R/\mathfrak{m}^i$ is Golod for all $1 \leqslant i \leqslant s$.

Here we focus on rings of socle degree 3. Our main result, Theorem (4.2), is a simultaneous converse to Propositions (3.1) and (3.3) in embedding dimension 3.

(3.3) **Proposition.** *Let $(R,\mathfrak{m})$ be an artinian Gorenstein local ring of embedding dimension at least 3 and socle degree 3. If $R$ has an exact zero divisor, then $R$ is compressed and Koszul, and the quotient ring $R/\mathfrak{m}^3$ is not Golod.*

**Proof.** Let $e$ denote the embedding dimension of $R$. By [6, thm. 3.3 and prop. 4.1] the existence of an exact zero divisor implies that $R$ is Koszul with Hilbert series $1+ez+ez^2+z^3$; hence $R$ is compressed. Further one has

$$\mathrm{P}^R_k(z) = \frac{1}{1-ez+ez^2-z^3} = \frac{(1+z)^e}{(1+z)^e(1-ez+ez^2-z^3)}\,.$$

Note that the denominator $(1+z)^e(1-ez+ez^2-z^3)$ is a polynomial of degree $e+3$. Let $Q$ be the regular ring of a minimal Cohen presentation of $R$; cf. (3.2). By [20, prop. 6.2] the ring $R/\mathfrak{m}^3$ is Golod if and only if one has

$$\mathrm{P}^R_k(z) = \frac{(1+z)^e}{1-z(\mathrm{P}^Q_R(z)-1)+z^{e+1}(z+1)}\,.$$

Since $\mathrm{P}^Q_R(z)$ is a polynomial of degree $e$, the denominator above is a polynomial of degree $e+2$. Thus $R/\mathfrak{m}^3$ is not Golod. □

To frame Propositions (3.1) and (3.3) we show how to produce a compressed artinian Gorenstein local ring $(T,\mathfrak{t})$ of socle degree 3 such that $T/\mathfrak{t}^3$ is not Golod, though $T$ is not complete intersection and does not have an exact zero divisor. See also Remark (5.3).

(3.4) **Proposition.** *Let $k$ be a field and $(R,\mathfrak{m})$ an artinian standard graded local $k$-algebra of embedding dimension $e \geqslant 3$ and socle degree 2. If $R$ is not Gorenstein and admits a non-zero minimal acyclic complex $F$ of finitely generated free modules, then the graded local $k$-algebra*

$$T = R \ltimes \Sigma\,\mathrm{Hom}_k(R,k) \quad \textit{with maximal ideal} \quad \mathfrak{t} = \mathfrak{m} \ltimes \Sigma\,\mathrm{Hom}_k(R,k)$$

*is Gorenstein and compressed with Hilbert series $1+(2e-1)z+(2e-1)z^2+z^3$. Furthermore, the following hold:*

(a) *$T$ is not complete intersection.*

(b) *If $R$ does not have an exact zero-divisor, then neither does $T$.*

(c) *If $\mathrm{H}_n(\mathrm{Hom}_R(F,R)) = 0$ holds for some $n$, then $T/\mathfrak{t}^3$ is not Golod.*

For a concrete example of a local $k$-algebra $(R,\mathfrak{m})$ that meets the assumptions in the Proposition—including those in parts (b) and (c)—see Christensen, Jorgensen, Rahmati, Striuli, and Wiegand [9, sec. 9].

**Proof.** From [10, thm. A] it is known that the Hilbert series of $R$ is $1+ez+(e-1)z^2$. As a graded $k$-vector space $T$ has the form $R \oplus \Sigma\,\mathrm{Hom}_k(R,k)$, so one has

$$\mathrm{H}_T(z) = \mathrm{H}_R(z) + z^3\,\mathrm{H}_R(z^{-1}) = 1+(2e-1)z+(2e-1)z^2+z^3.$$

Recall that $E = \mathrm{Hom}_k(R,k)$ is the injective envelope of $k$ over $R$. As a local ring, $T$ is the trivial extension of $R$ by $E$, so by [15, lem. in sec. 3] it is Gorenstein, and evidently it is compressed; cf. (3.2).

In the sequel, let $Q/I$ be a minimal Cohen presentation of $T$.

(a): If $T$ were complete intersection, then $I$ would be generated by $2e-1$ elements, but that is not possible as one has

$$h_Q(2) - h_T(2) = \binom{2e}{2} - (2e-1) = (e-1)(2e-1) > 2e-1\,.$$

(b): Assume towards a contradiction that $(x,\alpha)$ is an exact zero divisor in $T$ with annihilator generated by $(y,\beta)$. The element $(y,\beta)$ is also an exact zero divisor, called the complementary divisor; see [6, rmk. 1.1]. It follows from [6, prop. 4.1] that $(x,\alpha)$ and $(y,\beta)$ belong to $\mathfrak{t}\setminus\mathfrak{t}^2$. Hence $x$ and $y$ belong to $\mathfrak{m}\setminus\mathfrak{m}^2$ and, evidently, one has $xy=0$. Any element in $\mathfrak{m}^2 \ltimes 0$ annihilates $(x,\alpha)$ and is hence contained in the ideal generated by $(y,\beta)$. In particular, one has $\mathfrak{m}^2 = y\mathfrak{m}$ and by symmetry $\mathfrak{m}^2 = x\mathfrak{m}$. Now it follows from [9, lem. 4.3(c)] that $x$ and $y$ are exact zero-divisors in $R$, a contradiction.

(c) We argue that $T/\mathfrak{t}^3$ is not Golod by comparing two expressions for the Poincaré series of $T$. By a computation of Gulliksen [15, thm. 2] one has

$$\mathrm{P}^T_k(z) = \frac{\mathrm{P}^R_k(z)}{1 - z\,\mathrm{P}^R_E(z)}\,.$$

The standard isomorphisms $\mathrm{Tor}^R_*(k, \mathrm{Hom}_k(R,k)) \cong \mathrm{Hom}_k(\mathrm{Ext}^*_R(k,R),k)$ and [10, thm. A] yield:

$$\mathrm{P}^R_k(z) = \frac{1}{(1-z)(1-(e-1)z)} \qquad \text{and} \qquad \mathrm{P}^R_E(z) = \mathrm{I}_R(t) = \frac{e-1-z}{1-(e-1)z}\,.$$

Finally, a direct computation yields

$$\mathrm{P}^T_k(z) = \frac{(1+z)^{2e-1}}{(1-z)(1-2(e-1)z+z^2)(1+z)^{2e-1}}\,;$$

notice that the denominator has degree $2e+2$. On the other hand, the regular ring $Q$ has embedding dimension $2e-1$; in particular, $\mathrm{P}^Q_T(z)$ is a polynomial of degree $2e-1$. As in the proof of Proposition (3.3) above, it follows from [20, prop. 6.2] that the ring $T/\mathfrak{t}^3$ is Golod if and only if $\mathrm{P}^T_k(z)$ has the form $(1+z)^{2e-1}/d(z)$ where $d(z)$ is a polynomial of degree $2e+1$. □

## 4. Embedding dimension 3 and socle degree 3

Let $(R,\mathfrak{m},k)$ be a local ring of embedding dimension 3, let $\mathrm{K}^R$ be the Koszul complex on a minimal set of generators of $\mathfrak{m}$, and set $A = \mathrm{H}(\mathrm{K}^R)$. The Koszul complex is a differential graded algebra, and the product on $\mathrm{K}^R$ induces a graded-commutative $k$-algebra structure on $A$. As one has $A_{\geqslant 4} = 0$ it follows from Golod's original work [13] that $R$ is Golod if and only if $A$ has trivial multiplication, i.e. $A_{\geqslant 1} \cdot A_{\geqslant 1} = 0$. Moreover, it is known from work of Assmus [1, thm. 2.7] that $R$ is complete intersection if and only if $A$ is isomorphic to the exterior algebra on $A_1$.

There is a complete classification, due to Weyman [23] and Avramov et al. [4, 5], of artinian local rings of embedding dimension 3—even more generally of local rings of codepth $\leqslant 3$—based on multiplication in Koszul homology. For the precise statement of our main theorem, we need to recall one more class from this scheme: It is called **T**, and if $R$ belongs to this class one has $A_1 \cdot A_1 \neq 0 = A_1 \cdot A_2$; in particular $R$ is neither Golod nor complete intersection.

(4.1) **Remark.** An artinian local ring $(R,\mathfrak{m})$ of embedding dimension 2 and socle degree 2 is Gorenstein if and only if it is complete intersection if and only if it has an exact zero divisor if and only if every element in $\mathfrak{m}\setminus\mathfrak{m}^2$ is an exact zero divisor; see [9, rmk. (7.1)]. Such rings have Hilbert series $1+2z+z^2$, so they are compressed.

In Theorem (4.2) and Remark (5.1) we show that much of this behavior extends to embedding dimension and socle degree 3 but perhaps not further.

(4.2) **Theorem.** *Let $(R,\mathfrak{m})$ be an artinian Gorenstein local ring of embedding dimension 3 and socle degree 3. The following conditions are equivalent.*

($i$) *$R$ is complete intersection.*
($ii$) *$R$ is compressed and Koszul.*
($iii$) *$R$ has an exact zero divisor.*
($iv$) *$R/\mathfrak{m}^3$ belongs to the class* $\mathbf{T}$.
($v$) *$R/\mathfrak{m}^3$ is not Golod.*

To see that Theorem (1.1) follows from this statement, recall that a standard graded Koszul algebra is quadratic. Further, being Gorenstein, the ring $Q/I$ has a symmetric Hilbert series, i.e. it is $1+3z+3z^2+z^3$. Thus, if $I$ is quadratic, then it is minimally generated by 3 elements, which necessarily form a regular sequence.

**Proof.** Let $Q/I$ be a minimal Cohen presentation of $R$ and set $S=R/\mathfrak{m}^3$; cf. (3.2).

($i$)$\Longrightarrow$($iv$): If $R$ is complete intersection, then one has

$$\mathrm{P}_k^S(z)=\frac{1}{1-3z+2z^2-z^3}=\frac{(1+z)^2}{1-z-3z^2-z^5}$$

by (3.0.3), and that identifies $S$ as belonging to the class $\mathbf{T}$; see [4, thm. 2.1].

($iv$)$\Longrightarrow$($v$): Evident as rings of class $\mathbf{T}$ are not Golod.

($v$)$\Longrightarrow$($i$) by contraposition: If $R$ is not complete intersection, then [4, thm. 2.1] yields $\mathrm{P}_k^R(z)=(1+z)^2/g(z)$ with $g(z)=1-z-(\mu(I)-1)z^2-z^3+z^4$. By (3.0.2) one then has

$$\mathrm{P}_k^S(z)=\frac{(1+z)^2}{g(z)-z^2(1+z)^2}=\frac{(1+z)^2}{1-z-\mu(I)z^2-3z^3}\ ,$$

and, again by *loc. cit.*, that identifies $S$ as being Golod.

($i$)$\Longrightarrow$($iii$): It follows from (3.2) that $R$ has length at most $1+3+3+1=8$ with equality if and only if $R$ compressed. On the other hand, the length of $R$ is at least $2^3=8$ by [7, §7, prop. 7]. Thus $R$ is compressed with Hilbert series $1+3z+3z^2+z^3$; in particular, one has $(0:\mathfrak{m}^2)=\mathfrak{m}^2$; see [20, prop. 4.2(b)].

An application of [12, lem. 2.8] to the associated graded ring yields an element $\ell\in\mathfrak{m}\setminus\mathfrak{m}^2$ with $\ell\mathfrak{m}\neq\mathfrak{m}^2$. We argue that $\ell$ is annihilated by an element in $\mathfrak{m}\setminus\mathfrak{m}^2$. Assume towards a contradiction that $(0:\ell)\subseteq\mathfrak{m}^2$ holds. As $R$ Gorenstein, one now has $(\ell)\supseteq(0:\mathfrak{m}^2)=\mathfrak{m}^2$, and therefore, $\ell\mathfrak{m}=\mathfrak{m}^2$ by [6, rmk. 2.2(1)], which is a contradiction. Thus there exists an $\ell'\in\mathfrak{m}\setminus\mathfrak{m}^2$ with $\ell\ell'=0$. Notice that $\ell,\ell'\notin\mathfrak{m}^2=(0:\mathfrak{m}^2)$ implies $\ell\mathfrak{m}^2=\mathfrak{m}^3=\ell'\mathfrak{m}^2$.

To prove that $\ell$ and $\ell'$ are exact zero divisors, it now suffices by [6, prop. 4.1] to verify that $\mu(\ell\mathfrak{m})=2=\mu(\ell'\mathfrak{m})$; equivalently, that the linear maps $\mathfrak{m}/\mathfrak{m}^2\to\mathfrak{m}^2/\mathfrak{m}^3$ given by multiplication by $[\ell]_{\mathfrak{m}^2}$ and $[\ell']_{\mathfrak{m}^2}$ have kernels of rank 1. As $\ell\ell'=0$ the kernels have rank at least 1, and by symmetry it is sufficient to show that the rank is 1 for multiplication by $[\ell]_{\mathfrak{m}^2}$.

To this end, let $[\ell'']_{\mathfrak{m}^2} \neq 0$ be an element in the kernel of $l\colon \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{[\ell]} \mathfrak{m}^2/\mathfrak{m}^3$. That is, one has $\ell\ell'' \in \mathfrak{m}^3$ i.e. $\ell\ell'' = \ell u$ for some $u \in \mathfrak{m}^2$. The elements $\ell$, $\ell'$, and $\ell''$ lift to elements in $\mathfrak{n} \setminus \mathfrak{n}^2$, and $u$ lifts to an element in $\mathfrak{n}^2$; we denote these lifts by the same symbols. In $Q$ one now has $\ell\ell' \in I$ and $\ell(\ell'' - u) \in I$. Let $f, g, h \in \mathfrak{n}^2$ be a regular sequence that generates $I$ and write

$$\ell\ell' = a'f + b'g + c'h \qquad \text{and} \qquad \ell(\ell'' - u) = a''f + b''g + c''h$$

with coefficients $a', \dots, c''$ in $Q$. As the associated graded ring of $Q$ is a domain, one has $v_Q(\ell\ell') = v_Q(\ell) + v_Q(\ell') = 1 + 1 = 2$. That is, $\ell\ell'$ is in $\mathfrak{n}^2 \setminus \mathfrak{n}^3$, so we may without loss of generality assume that $c'$ is a unit. After cross multiplication by $\ell'$ and $\ell'' - u$ and elimination of $\ell\ell'(\ell'' - u)$ one gets $((\ell'' - u)c' - \ell'c'')h \in (f, g)$. As $f, g, h$ is a regular sequence, this implies $((\ell'' - u)c' - \ell'c'') \in (f, g) \subseteq \mathfrak{n}^2$, and since $u$ is in $\mathfrak{n}^2$ it further implies $\ell''c' - \ell'c'' \in \mathfrak{n}^2$. Recall that $c'$ is a unit. If $c''$ were in $\mathfrak{n}$, one would have $\ell'' \in \mathfrak{n}^2$, contrary to the assumptions. Thus $c''$ is a unit, whence $\ell'$ and $\ell''$ are linearly dependent mod $\mathfrak{n}^2$. That is, $[\ell']_{\mathfrak{m}^2}$ spans the kernel of $l$.

*(iii)*$\Longrightarrow$*(ii)*: By Proposition (3.3).

*(ii)*$\Longrightarrow$*(i)*: One has $\mathrm{P}^R_k(z) = 1/\,\mathrm{H}_R(-z) = 1/(1 - 3z + 3z^2 - z^3) = 1/(1 - z)^3$, which means that $R$ is complete intersection; see [4, thm. 2.1]. □

While [20, prop. 6.3] is a statement about proper quotients $R/\mathfrak{m}^i$ of all compressed Gorenstein rings of socle degree not 3, there is no uniform behavior of those of socle degree 3.

(4.3) **Example.** Let $k$ be a field. By a result of Buchsbaum and Eisenbud [8, thm. 2.1], the defining ideal of a Gorenstein ring $R = k[[x, y, z]]/I$ is generated by the sub-maximal Pfaffians of an odd-sized skew-symmetric matrix.

The ideal generated by the $4 \times 4$ Pfaffians of the matrix

$$\begin{pmatrix} 0 & x+y & 0 & 0 & y \\ -x-y & 0 & 0 & y^2+z^2 & yz \\ 0 & 0 & 0 & x+z & z \\ 0 & -y^2-z^2 & -x-z & 0 & x \\ -y & -yz & -z & -x & 0 \end{pmatrix}$$

has the form

$$I = (xz + yz, xy + yz, x^2 - yz, yz^2 + z^3, y^3 - z^3)\,.$$

It is straightforward to verify that a graded basis for $R$ is

$$1\,; \quad x, y, z\,; \quad y^2, yz, z^2\,; \quad z^3\,.$$

Thus $R$ is a compressed artinian Gorenstein ring of socle degree 3. It is not complete intersection, so by Theorem (4.2) it does not have exact zero-divisors.

## 5. Remarks on embedding dimension 4

The proof of the implication *(i)* ⇒ *(iii)* in Theorem (4.2) relies on $R$ being compressed, so it seems fitting to record the following remark.

(5.1) **Remark.** A compressed artinian Gorenstein ring $(R, \mathfrak{m})$ of embedding dimension $e \geqslant 4$ cannot be complete intersection. Indeed, let $Q/I$ be a minimal Cohen presentation of $R$ and let $s$ denote the socle degree. As $R$ is compressed, the initial degree of $I$ is $t = \min_i\{h_Q(s-i) < h_Q(i)\}$, and by [20, prop. 4.2] one has $t = \lceil \frac{s+1}{2} \rceil$. A straightforward computation yields

$$\mu(I) \geqslant h_Q(t) - h_Q(s-t) = \begin{cases} \binom{e-2+t}{e-2} & \text{for odd } s \\ \binom{e-2+t}{e-2} + \binom{e-3+t}{e-2} & \text{for even } s. \end{cases}$$

By minimality of $Q/I$ one has $t \geqslant 2$ and hence $\mu(I) \geqslant \binom{e}{e-2} = \frac{e(e-1)}{2} > e$.

For artinian Gorenstein local rings of embedding dimension 4—and more generally for Gorenstein local rings of codepth 4—there is a classification based on multiplication in Koszul homology. It predates the classification of local rings of codepth 3 and was achieved by Kustin and Miller [18]; for simplicity we refer here to Avramov's exposition in [2]. In addition to the class of complete intersections, the classification scheme has three classes one of which is called **GGO** in [2].

(5.2) **Proposition.** *An artinian Gorenstein local ring* $(R, \mathfrak{m})$ *of embedding dimension* 4 *and socle degree* $s$ *belongs to the class* **GGO** *if and only if* $R/\mathfrak{m}^s$ *is Golod.*

**Proof.** Let $k$ denote the residue field of $R$. The Poincaré series of $R$ has the form $(1+z)^4/d(z)$, where the polynomial $d(z)$ depends on the class of $R$ as proved by Jacobsson, Kustin, and Miller [17]. By (3.0.2) one has

$$\mathrm{P}_k^{R/\mathfrak{m}^s}(z) = \frac{(1+z)^4}{d(z) - z^2(1+z)^4} \,.$$

For $R/\mathfrak{m}^s$ to be Golod, the denominator $D(z) = d(z) - z^2(1+z)^4$ has to be a polynomial of degree 5, see (3.0.1), but if $R$ is not of class **GGO**, then $d(z)$ and hence $D(z)$ has degree at least 7; see [2, thm. (3.5)].

It remains to prove that $R/\mathfrak{m}^s$ is Golod if $R$ is of class **GGO**. For a ring $R$ of this class, one gets from [2, thm. (3.5)] the expression

$$\begin{aligned} D(z) &= (1+z)^2(1 - 2z - (h-3)z^2 - 2z^3 + z^4) - z^2(1+z)^4 \\ &= (1+z)^2(1 - 2z - (h-2)z^2 - 4z^3) \\ &= 1 - (h+1)z^2 - 2(h+1)z^3 - (h+6)z^4 - 4z^5 \,, \end{aligned}$$

where $h$ denotes the minimal number of generators of the defining ideal in a minimal Cohen presentation of $R$ (in [2] this number is called $l+1$). Set $h_j = \dim_k \mathrm{H}_j(\mathrm{K}^R)$; as $R$ is Gorenstein one has $h_0 = 1 = h_4$, $h_1 = h = h_3$ and $h_2 = 2h-2$. By [19, thm. 1] there is an isomorphism of $k$-algebras

$$\mathrm{H}(K^{R/\mathfrak{m}^s}) \cong \mathrm{H}(K^R)/\,\mathrm{H}_4(K^R) \ltimes (\Sigma \textstyle\bigwedge k^4)/(\Sigma \textstyle\bigwedge^4 k^4) \,.$$

In particular, one gets

$$\begin{aligned} \dim_k \mathrm{H}_1(\mathrm{K}^{R/\mathfrak{m}^s}) &= h_1 + 1 = h+1 \\ \dim_k \mathrm{H}_2(\mathrm{K}^{R/\mathfrak{m}^s}) &= h_2 + 4 = 2(h+1) \\ \dim_k \mathrm{H}_3(\mathrm{K}^{R/\mathfrak{m}^s}) &= h_3 + 6 = h+6 \\ \dim_k \mathrm{H}_4(\mathrm{K}^{R/\mathfrak{m}^s}) &= 4 \,, \end{aligned}$$

and comparison of the expression for $D(z)$ to (3.0.1) shows that $R/\mathfrak{m}^s$ is Golod. □

(5.3) **Remark.** Let $(R, \mathfrak{m}, k)$ be an artinian Gorenstein local ring of embedding dimension 4 and socle degree $s$. In the terminology of [2], $R$ is complete intersection or of class **GGO**, **GT**, or $\mathbf{GH}(p)$; here the parameter $p$ is between 1 and $h-1$ where $h$, as in the proof above, is the rank of $\mathrm{H}_1(K^R)$ or, equivalently, the minimal number of generators of the defining ideal in a minimal Cohen presentation of $R$.

Examples of rings of class **GT** and **GH** are provided in [18, prop. (2.7) and (2.8)]; they are examples of local Gorenstein rings that are not complete intersection and have $R/\mathfrak{m}^s$ not Golod, compare Proposition (3.1).

If $R$ has socle degree 3 and $R$ has an exact zero divisor, then [6] yields

$$\mathrm{P}_k^R(z) = \frac{1}{1-4z+4z^2-z^3} = \frac{(1+z)^4}{(1+z)^2(1-2z-3z^2+3z^3+2z^4-z^5)}\,,$$

which by [2, thm. 3.5] identifies $R$ as being of class $\mathbf{GH}(5)$ where $5 = h-1$.

Example (5.4) exhibits a concrete local ring $R$ of socle degree 3 that is not complete intersection and does not have an exact zero divisor but such that $R/\mathfrak{m}^3$ is not Golod, compare Proposition (3.3).

(5.4) **Example.** Let $k$ be a field and set $Q = k[\![w,x,y,z]\!]$. The ideal

$$I = (w^2+xy, wx+xz, wz, y^2+xz, yz, z^2, x^3+x^2z)$$

defines a $k$-algebra with basis

$$1\,; \quad w,x,y,z\,; \quad wy, x^2, xy, xz\,; \quad x^2z\,;$$

in particular, it has socle degree 3. Proceeding as in [11] one can use Macaulay 2 [14] to verify that $Q/I$ is a Gorenstein ring of class **GT**.

## Acknowledgment

We thank Pedro Macias Marques for discussions related to the material in this paper, and we thank the anonymous referee for pertinent questions that prompted us to add Section 5.

## References

[1] Edward F. Assmus, Jr., *On the homology of local rings*, Illinois J. Math. **3** (1959), 187–199. MR0103907

[2] Luchezar L. Avramov, *Homological asymptotics of modules over local rings*, Commutative algebra (Berkeley, CA, 1987), Math. Sci. Res. Inst. Publ., vol. 15, Springer, New York, 1989, pp. 33–62. MR1015512

[3] Luchezar L. Avramov, *Infinite free resolutions*, Six lectures on commutative algebra (Bellaterra, 1996), Progr. Math., vol. 166, Birkhäuser, Basel, 1998, pp. 1–118. MR1648664

[4] Luchezar L. Avramov, *A cohomological study of local rings of embedding codepth 3*, J. Pure Appl. Algebra **216** (2012), no. 11, 2489–2506. MR2927181

[5] Luchezar L. Avramov, Andrew R. Kustin, and Matthew Miller, *Poincaré series of modules over local rings of small embedding codepth or small linking number*, J. Algebra **118** (1988), no. 1, 162–204. MR0961334

[6] Inês Bonacho Dos Anjos Henriques and Liana M. Şega, *Free resolutions over short Gorenstein local rings*, Math. Z. **267** (2011), no. 3-4, 645–663. MR2776052

[7] Nicolas Bourbaki, *Éléments de mathématique*, Masson, Paris, 1983, Algèbre commutative. Chapitre 8. Dimension. Chapitre 9. Anneaux locaux noethériens complets. MR0722608

[8] David A. Buchsbaum and David Eisenbud, *Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension* 3, Amer. J. Math. **99** (1977), no. 3, 447–485. MR0453723

[9] Lars Winther Christensen, David A. Jorgensen, Hamidreza Rahmati, Janet Striuli, and Roger Wiegand, *Brauer-Thrall for totally reflexive modules*, J. Algebra **350** (2012), 340–373. MR2859892
[10] Lars Winther Christensen and Oana Veliche, *Acyclicity over local rings with radical cube zero*, Illinois J. Math. **51** (2007), no. 4, 1439–1454. MR2417436
[11] Lars Winther Christensen and Oana Veliche, *Local rings of embedding codepth 3: a classification algorithm*, J. Softw. Algebra Geom. **6** (2014), 1–8. MR3338667
[12] Aldo Conca, Maria Evelina Rossi, and Giuseppe Valla, *Gröbner flags and Gorenstein algebras*, Compositio Math. **129** (2001), no. 1, 95–121. MR1856025
[13] Evgeniy S. Golod, *Homologies of some local rings*, Dokl. Akad. Nauk SSSR **144** (1962), 479–482. MR0138667
[14] Daniel R. Grayson and Michael E. Stillman, *Macaulay 2, a software system for research in algebraic geometry*, available at http://www.math.uiuc.edu/Macaulay2/.
[15] Tor H. Gulliksen, *Massey operations and the Poincaré series of certain local rings*, J. Algebra **22** (1972), 223–232. MR0306190
[16] Anthony Iarrobino, *Compressed algebras: Artin algebras having given socle degrees and maximal length*, Trans. Amer. Math. Soc. **285** (1984), no. 1, 337–378. MR0748843
[17] Carl Jacobsson, Andrew R. Kustin, and Matthew Miller, *The Poincaré series of a codimension four Gorenstein ring is rational*, J. Pure Appl. Algebra **38** (1985), no. 2-3, 255–275. MR0814181
[18] Andrew R. Kustin and Matthew Miller, *Classification of the Tor-algebras of codimension four Gorenstein local rings*, Math. Z. **190** (1985), no. 3, 341–355. MR0806892
[19] Gerson L. Levin and Luchezar L. Avramov, *Factoring out the socle of a Gorenstein ring*, J. Algebra **55** (1978), no. 1, 74–83. MR0515760
[20] Maria Evelina Rossi and Liana M. Şega, *Poincaré series of modules over compressed Gorenstein local rings*, Adv. Math. **259** (2014), 421–447. MR3197663
[21] Günter Scheja, *Über die Bettizahlen lokaler Ringe*, Math. Ann. **155** (1964), 155–172. MR0162819
[22] Liana M. Şega, *Homological properties of powers of the maximal ideal of a local ring*, J. Algebra **241** (2001), no. 2, 827–858. MR1843329
[23] Jerzy Weyman, *On the structure of free resolutions of length* 3, J. Algebra **126** (1989), no. 1, 1–33. MR1023284

Texas Tech University, Lubbock, TX 79409, U.S.A.
*E-mail address*: lars.w.christensen@ttu.edu
*URL*: http://www.math.ttu.edu/~lchriste

Northeastern University, Boston, MA 02115, U.S.A.
*E-mail address*: o.veliche@northeastern.edu