Def^n Let V be a set of cardinality $n+1$. The n-simplex on V is $\mathcal{P}(V)$.

Ex $V = \{v_1, v_2, v_3\}$

2-simplex on V

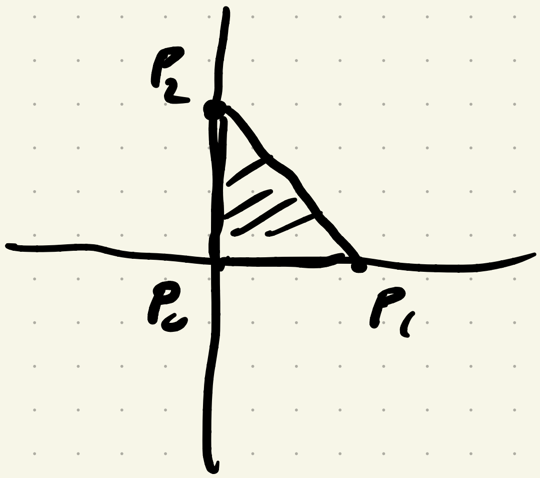
$$\{\{v_1, v_2, v_3\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_1\}, \{v_2\}, \{v_3\}, \emptyset\}$$

Def^n Let $P_0, \ldots, P_n$ be points in $\mathbb{R}^n$ s/t $P_0 - P_1$, $P_0 - P_2$, $\ldots$ $P_0 - P_n$ are lin indep.

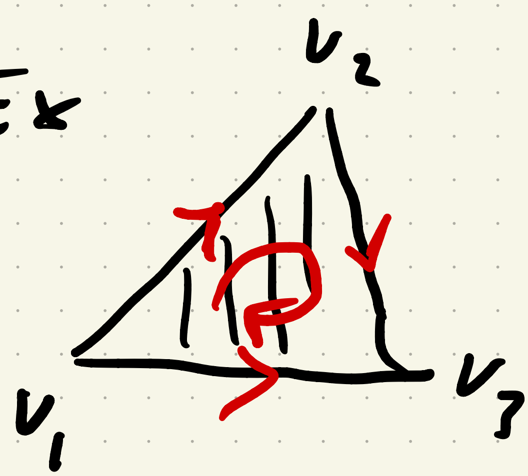
The convex combinations $P_0, \ldots, P_n$ is called the geometric n-simplex

The standard unit n-simplex is given by the unit coordinate vectors in $\mathbb{R}^n$.

Ex $n = 2$

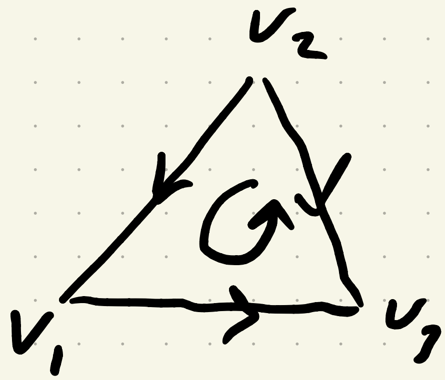


Ex

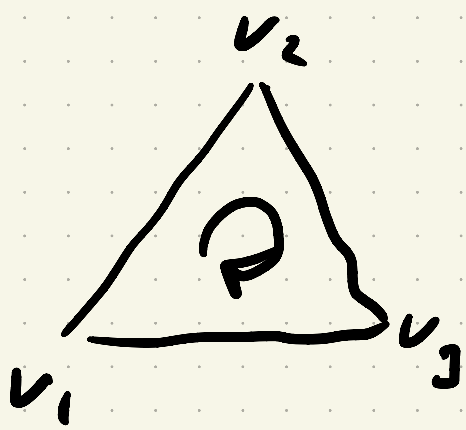


$V_1 < V_2 < V_3$

$v_2 < v_1 < v_3$



$v_2 < v_3 < v_1$



Orientations are equivalent if they differ by an even permutation

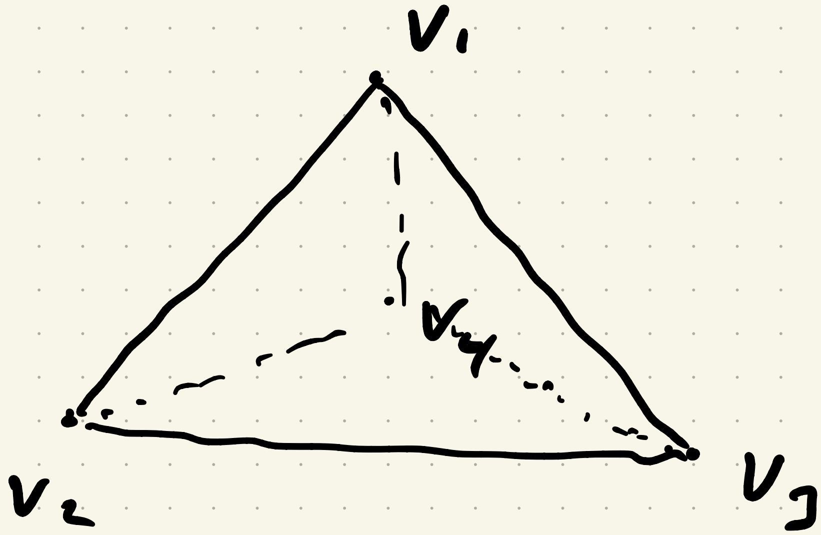
Def^n A simplicial cx Δ on a vertex set V is a subset of $\mathcal{P}(V)$ s/t

- $v \in \Delta$, $\forall v \in V$
- If $\tau \in \Delta$ and $\sigma \subseteq \tau$, then $\sigma \in \Delta$

The dim of Δ is the dim of the largest simplex it contains. A $\tau \in \Delta$ with $|\tau| = i+1$ is called an i-face.

<u>Ex</u> $V = \{v_1, v_2, v_3, v_4\}$

$$\Delta = \mathcal{P}(\{v_1, v_2, v_3\}) \cup \mathcal{P}(\{v_1, v_2, v_4\})$$
$$\cup \, \mathcal{P}(v_1, v_3, v_4) \cup \mathcal{P}(\{v_2, v_3\, v_4\})$$



Given a ring R

$C_i(\Delta) = C_i$ is the module generated by the oriented i-faces subject to

$$\{v_{j_0}, \dots, v_{j_i}\} = (-1)^{\operatorname{sgn}(\pi)} \{v_{j_{\sigma(0)}}, \dots, v_{j_{\sigma(i)}}\}$$

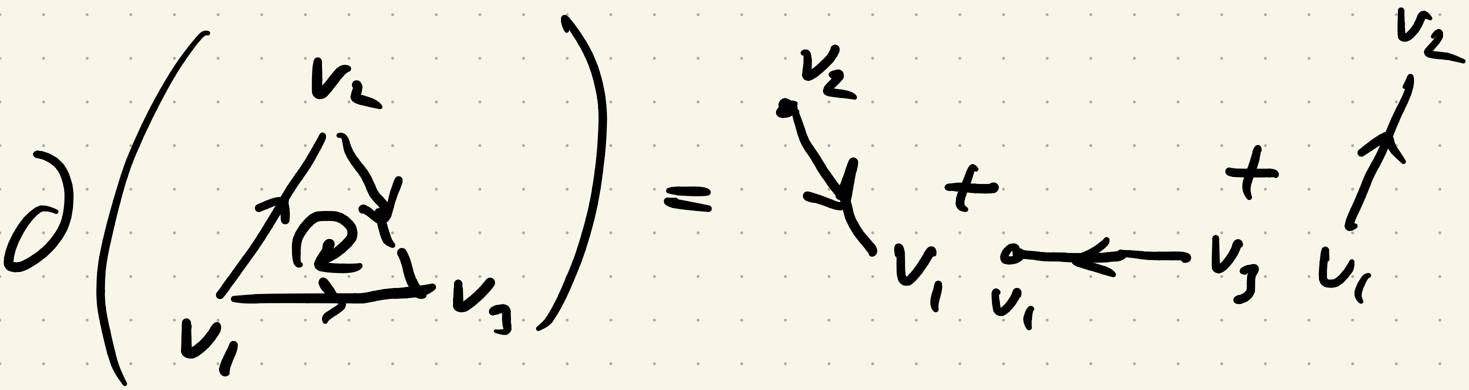
for $\sigma \in S_i$

For tetrahedron

$$0 \to R^4 \to R^4 \xrightarrow{\partial} R^4 \to R \to 0$$

$3 \qquad 2 \qquad 1 \qquad 0 \qquad -1 \qquad -2$

$$\partial(\{v_{i_0},\dots,v_{i_n}\}) = \sum_{j=0}^{n} (-1)^j \cdot \{v_{i_0},\dots,\check{v}_{i_j},\dots,v_{i_n}\}$$



claim $\partial^2 = 0$

Pf: $\partial^2(\{v_0,\dots,v_n\}) =$

$$\partial\left(\sum_{j=0}^{n}(-1)^j\{v_0,\dots,\check{v}_j \dots, v_n\}\right) =$$

$$= \sum_{i<j}(-1)^{i+j}\{v_0,\dots,\check{v}_i,\dots,\check{v}_j,\dots v_n\}$$

$$\sum_{j<i} (-1)^{j+i-1} \{v_0, \ldots, \check{v}_j, \ldots, \check{v}_i, \ldots, v_n\}$$

$$=$$

<u>Def</u> Given a simplicial cx Δ the homology of Δ, written $H_*(\Delta)$ is the homology of $C_*(\Delta)_{\geq 0}$.
The reduced homology of Δ, $\tilde{H}_*(\Delta)$ is the homology of all of $C(\Delta)$

Claim: $\operatorname{rank} H_0(\Delta) = \operatorname{rank} \tilde{H}(\Delta) + 1$

$$C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{[1\,1\,1\,1]} C_{-1}$$

Homology in deg 0:

$$H_0(\Delta) = \frac{C_0}{\operatorname{im} \partial_1}$$

$\operatorname{rank} C_0 = n$

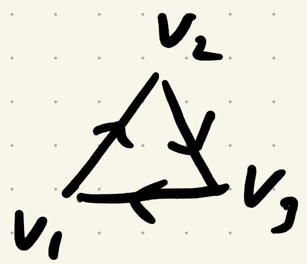
$$\ker \partial_0 = \operatorname{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \\ \vdots \end{pmatrix} \cdots \begin{pmatrix} -1 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right\}$$

$\operatorname{rank} \ker \partial_0 = n-1$

<u>Remark</u> $\tilde{H}_0(\Delta) = 0 \Leftrightarrow$ Δ is connected.

I.e. $|\Delta|$ the geometric realization of Δ is connected $\mathbb{R}^n$

$R = \mathbb{Q}$

$\Delta =$ 

Basis for C_0 : $\{v_1\}, \{v_2\}, \{v_3\}$

Basis for C_1 : $\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_1\}$

$$0 \longrightarrow \underset{C_1}{\mathbb{Q}^3} \xrightarrow{\begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}} \underset{C_0}{\mathbb{Q}^3} \xrightarrow{[1\,1\,1]} \underset{C_{-1}}{\mathbb{Q}}$$

$$\partial([v_1, v_2]) = [v_2] - [v_1]$$

$$\tilde{H}_0(\Delta) = 0$$

$$\mathrm{rk}(H_1(\Delta)) = 1$$

$$\begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$