Hilbert's Syzygy Thm

Let M be a f.g. graded module over $k[x_0, \dots, x_n]$. There is a graded free resolution

$$0 \to F_{n+1} \to \cdots \to F_1 \to F_0 \to M \to$$

Def^n The resolution F is called **minimal** if the graded maps are represented by matrices with no constant terms.

Exa $I = \langle x^2 \rangle$

$$0 \to \underbrace{R(-2) \xrightarrow{x^2} R}_{F} \longrightarrow R/\langle x^2 \rangle \to 0$$

$$0 \to R(-3) \xrightarrow{\begin{bmatrix} y \\ -1 \end{bmatrix}} \begin{matrix} R(-2) \\ \oplus \\ R(-3) \end{matrix} \xrightarrow{\begin{bmatrix} x^2 & yx^2 \end{bmatrix}} R \longrightarrow R/\langle x^2 \rangle \to 0$$

Exa.

$$\cdots \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \cdots$$

$$\longrightarrow \begin{matrix} R \\ \oplus \\ F_i \end{matrix} \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & d_i \end{bmatrix}} \begin{matrix} R \\ \oplus \\ F_{i-1} \end{matrix} \longrightarrow$$

$$0 \to R \xrightarrow{1} R \to 0$$

$$0 \to F_{n+1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

$$F_j = \bigoplus_{k=1}^{r_j} R(-a_{jk})$$

$$HP(M,i) = \sum_{j=0}^{n+1} (-1)^j HP(F_j, i)$$

$$= \sum_{j=0}^{n+1} (-1)^j \sum_{k=1}^{r_j} \binom{n+i-a_{jk}}{n}$$

$$HS(R(-a),t) = t^a HS(R,t)$$

$$= \frac{t^a}{(1-t)^{n+1}}$$

$$HS(F_j,t) = \frac{\sum_{k=1}^{r_n} t^{a_{jk}}}{(1-t)^{n+1}}$$

$$HS(M,t) = \sum_{j=0}^{n+1} (-1)^j HS(F_j,t)$$

$$= \frac{P(t,t^{-1})}{(1-t)^{n+1}}$$

Recall

$$HP(R/I, i) = \frac{a_n}{n!} i^n + \cdots$$

$n = \dim V(I)$

$a_n = \text{degree } V(I)$

Claim (Exc. 3.3.1). Let $I \subseteq R$ be homogeneous and $f \in R$ homogeneous of degree d.

Exact sequence:

$$0 \to R(-d)/I:f \to R/I \to R/\langle I, f\rangle \to 0$$

If $d = 1$ and f is not a zero-divisor on R/I, then

$I : f = I$

$$\therefore \; HP(R/\langle I, f\rangle, i)$$

$$HP(R/I, i) - HP(R/I, i-1)$$

$$= \frac{a_n}{n!} i^n + \frac{a_{n-1}}{(n-1)!} i^{n-1} + \cdots$$

$$- \left(\frac{a_n}{n!} (i-1)^n + \frac{a_{n-1}}{(n-1)!} (i-1)^{n-1} + \cdots \right.$$

$$= -\frac{a_n}{n!} \left(-n i^{n-1} \right) + \cdots$$

$$= \frac{a_n}{(n-1)!} i^{n-1} + \cdots$$

We can't always find regular elements!

$I = \bigcap_{i=1}^{n} Q_i$, if $\sqrt{Q_k} = \mathfrak{m} =$ irrelevant max'l ideal $\langle x_0, \ldots, x_n \rangle$

For $f \in Q_k$

$$I : f = \bigcap Q_i : f = \bigcap (Q_i : f) \supset I.$$

<u>Prime Avoidance.</u> If $I \subseteq \bigcup_{i=1}^n P_i$ with P_i prime, then $I \subseteq P_i$ for some i.

<u>Pf</u> By contraposition and induction on n.

(*) $\forall i : I \not\subseteq P_i \Rightarrow I \not\subseteq \bigcup_{i=1}^n P_i$

<u>$n=1$</u> clear!

$n > 1$ and assume that (*) holds for $n-1$. Assume towards a contradiction that

$$\forall i : I \not\subseteq P_i \text{ and } I \subseteq \bigcup_{i=1}^n P_i$$

By hypothesis

$$I \nsubseteq \bigcup_{j \neq i} P_j$$

$$\therefore \quad \exists\, x_i \in I \setminus \bigcup_{i \neq j} P_j$$

so $x_i \in P_i$ as

Set

$$x = \sum_{i=1}^{n} x_1 x_2 \cdots \check{x}_i \cdots x_n \in I$$

Goal $x \notin P_i$ for all i

$$x_1 \cdots \check{x}_i \cdots x_n \notin P_i$$

For $i \neq k$

$$x_1 \cdots \check{x}_k \cdots x_n \in P_i$$

$$\therefore \quad x \notin P_i \qquad \square$$