$R = k[x, y, z]$

$I = \langle x^3 + y^3 + z^3 \rangle$

$$0 \to R(-3) \xrightarrow{x^3+y^3+z^3} R \longrightarrow \frac{R}{I} \to 0$$

$$HF(R/I, i) = HP(R, i) - HP(R(-3), i)$$

$$= \binom{2+i}{2} - \binom{2+(i-3)}{2}$$

$$= \binom{2+i}{2} - \binom{i-1}{2}$$

$$= \binom{2+i}{2} - \binom{1+i}{2} + \binom{1+i}{2} - \binom{i-1}{2}$$

$$\left[\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1} \right]$$

$$\binom{1+i}{1} + \binom{1+i}{2} - \binom{i}{2} + \binom{i}{2} - \binom{i-1}{2}$$

$$= (1+i) + \binom{i}{1} + \binom{i-1}{1}$$

$$= 3i$$

$$= 3\binom{i+1}{1} - 3 \quad [3P_1 - 3P_0]$$

$$0 \to R(-4) \xrightarrow{\begin{bmatrix} x^2+y^2+z^2 \\ -x \end{bmatrix}} \begin{matrix} R(-1) \\ \oplus \\ R(-3) \end{matrix} \xrightarrow{\begin{bmatrix} x & x^2+y^2+z^2 \end{bmatrix}} R \to \frac{R}{\langle x, x^2+y^2+z^2 \rangle} \to 0$$

$$\begin{bmatrix} a \\ b \end{bmatrix} \mapsto ax + b(x^2+y^2+z^2) = 0$$

$$ax = -b(x^2+y^2+z^2)$$

$$= -b'x(x^2+y^2+z^2)$$

$$a'(x^2+y^2+z^2)x$$

$$\therefore a' = -b'$$

$$HP(R/\langle x, x^2+y^2+z^2\rangle) = HP(R)$$

$$- HP(R(-1), i) - HP(R(-3), i)$$

$$+ HP(R(-4), i) =$$

$$\binom{2+i}{2} - \binom{1+i}{2} - \binom{i-1}{2} + \binom{i-2}{2}$$

$$\binom{1+i}{1} - \left[\binom{i-1}{2} - \binom{i-2}{2}\right]$$

$$= 1+i - (i-2) = 3 \quad (3P_0)$$

In M2 lingo:

$$P_k(i) = HP(k[x_0,\dots,x_k], i)$$

$$\therefore \; P_0(i) = HP(k[x_0], i) = 1$$

— x —

<u>Exc. 3.2.1</u>

$I = \langle f, g \rangle \subseteq k[x,y,z] =: R$

f, g homogeneous w/o common factors. $\deg f = d$, $\deg g = e$

$$0 \to R(-d-e) \xrightarrow{\begin{bmatrix} g \\ -f \end{bmatrix}} \begin{matrix} R(-d) \\ \oplus \\ R(-e) \end{matrix} \xrightarrow{[f\ g]} R \longrightarrow \frac{R}{I} \to 0$$

$HP(R/I, i) =$

$$\binom{2+i}{2} - \binom{2+i-d}{2} - \binom{2+i-e}{2} + \binom{2+i-e-f}{2}$$

$$= \binom{2+i}{2} - \binom{2+i-d}{2} - \left[\binom{2+i-e}{2} - \binom{2+i-d-e}{2}\right]$$

$$= di + \sum_{j=1}^{d} 2-j - \left[d(i-e) + \sum_{j=1}^{d} 2-j\right]$$

———— × ————

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \binom{n+1}{2}$$

———— × ——

$$= \sum_{k=1}^{1+i} k - \sum_{k=1}^{1+i-d} - [\qquad]$$

$$= \sum_{k=2+i-d}^{1+i} k = (2+i-d) + (2+i-d+1) \cdots$$

—— × ——

$$= de.$$

Lemma. Let $0 \to M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \to 0$ be exact. The module M is noetherian if and only if M' and M'' are noetherian.

<u>Pf</u> "Only if":

A sequence

(*) $M_0' \subseteq M_1' \subseteq \cdots$ in M'

yields a seq. of isomorphic modules

$$\varphi(M_0') \subseteq \varphi(M_1') \subseteq \cdots \text{ in } M$$

which stabilizes, so (*) stabilizes.

Similarly,

(**) $M_0'' \subseteq M_1'' \subseteq \cdots$ in $M'' \cong M/M'$

Yields a sequence of submodules of M (that contain $\varphi(M')$.

"If". $M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$ in M

$\psi(M_i) = \psi(M_{i+1})$, $\forall i \gg 0$

And

$M_i \cap \varphi(M') = M_{i+1} \cap \varphi(M')$ $i \gg 0$

For $i \gg 0$ $m \in M_{i+1}$

$$\psi(m) \in \psi(M_i)$$

$$\therefore \psi(m) = \psi(\hat{m}), \hat{m} \in M_i$$

$$\psi(m - \hat{m}) = 0$$

$$m - \hat{m} \in \varphi(M') \cap M_{i+1}$$

$$m - \hat{m} = \hat{\hat{m}} \in \varphi(M') \cap M_i$$

$$m = \hat{m} + \hat{\hat{m}} \in M_i \quad \blacksquare$$

Prop. Let R be noetherian. Every f.g. free R-module is Noetherian.

Pf Choose a basis $e_1, \dots, e_n$ and induct on n using the lemma:

$$0 \to R\langle e_n \rangle \to R\langle e_1, \dots, e_n \rangle \twoheadrightarrow R\langle e_1, \dots, e_{n-1} \rangle$$

Consequence: If M f.g. over a noetherian ring then one can construct a free resolution:

$$\cdots \longrightarrow R^{b_1} \longrightarrow R^{b_0} \longrightarrow M \longrightarrow 0$$

$$R^{b_1} \searrow K \nearrow R^{b_0}$$

$K \nwarrow$ f.g.

Exa. $\mathbb{Z}_4$ local ring with max'l ideal $2\mathbb{Z}_4$.

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{2} & \mathbb{Z}_4 & & \xrightarrow{2} & \mathbb{Z}_4 & \xrightarrow{2} & 2\mathbb{Z}_4 & \to 0 \\
 & & & \searrow & & \nearrow & & & \\
 & & & & 2\mathbb{Z}_4 & & & &
\end{array}$$

Hilbert's Syzygy Thm.

Every f.g. graded module M over $k[x_1,\dots,x_n]$ has a graded free resolution

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$$

where the modules F_j are f.g. free modules.