LIVING WITH A CONCEPT: LEAST UPPER BOUNDS

MATH 3310 SPRING 2019. PROJECT B.

Instructions: Over the course of four days, a new question related to partitions will appear on Blackboard every day; your solutions are due at 11.59 pm on Monday–Thursday.

> *When dealing with a subset of $\mathbb{R}$, it is often useful to look for a number greater than or equal to every number in the subset.*

Day 1. Let A be a subset of $\mathbb{R}$. Define an **upper bound** for A to be an element $x \in \mathbb{R}$ such that $a \leq x$ holds for all $a \in A$. For each of the following sets, decide if it has an upper bound and if so identify one.

$$
\begin{aligned}
A &= \{-1, 0, 1\} \\
B &= \{n \in \mathbb{Z} : n \leq 7\} \\
C &= (-5, 10) \\
D &= [5, \infty) \\
E &= [2.5, 3]
\end{aligned}
$$

Day 2. Define a **least upper bound** for a set $A \subset \mathbb{R}$ to be an upper bound u for A that satisfies $u \leq x$ for every upper bound x for A. Think of it as, "Among all the numbers that are larger than everything in A, a least upper bound is less than or equal to all of them". Using the definition from yesterday, determine if the sets below have an upper bound and, if so, identify a least upper bound.

$$
\begin{aligned}
F &= (0, \infty) \\
G &= \{-8000, 0, 1, 2, 3, 100\} \\
H &= \{n \in \mathbb{Z} : n \leq \tfrac{3}{2}\} \\
I &= (4, 14) \\
J &= (-\infty, 3]
\end{aligned}
$$

Day 3. Mathematicians typically prefer working with things that are unique, meaning there is only one of those things. Remember that a **least upper bound** for a set A is a number greater than or equal to everything in A, but less than or equal to every upper bound.

For today's task, prove that a least upper bound is unique and can, therfore, if it exists be called *the* least upper bound. **Hint:** Assume that you have two least upper bounds u_1 and u_2 for the same set. Using the definition of a least upper bound, how do u_1 and u_2 compare?

Day 4. True or false: For a set $A \subset \mathbb{R}$, the least upper bound of A must be an element of A. Please explain your answer; it does not need to be a formal proof.