Name: ______________________________

MATH 3310
FINAL EXAM
SPRING 2018

DEPARTMENT OF MATH AND STAT
TEXAS TECH UNIVERSITY

Part I

Solve problems 1–10 below; they count for 4% each. Your answers must be written on this sheet of paper. No aids are allowed on this part of the test. When you have turned in this part, you may use books and notes to solve the problems on Part II. For full credit, you must show complete, correct, legible work. Read carefully before you start working.

1. For $z \in \mathbb{Z}$ let $[z]$ denote the equivalence class of z modulo 11.
   (a) Determine the product $[3][7]$.
   (b) Find an integer a with $[6][a] = [1]$.

2. Which of the following functions defined on $\mathbb{R}$ are 1-to-1 (injective)? Mark them with a ✓.
   (a) e^x
   (b) $\cos x$
   (c) x^2
   (d) x^3

3. Which of the following functions are onto (surjective)? Mark them with a ✓.
   (a) $f\colon (0,\infty) \to \mathbb{R}$ given by $f(x) = \ln x$
   (b) $f\colon (\frac{\pi}{2}, \frac{3\pi}{2}) \to \mathbb{R}$ given by $f(x) = \tan x$
   (c) $f\colon (0,\infty) \to \mathbb{R}$ given by $f(x) = x^2$
   (d) $f\colon \mathbb{R} \to (0,\infty)$ given by $f(x) = 2^x$

4. On the set $A = \{a, b, c, d\}$ consider the relation
$$R = \{(a,a), (b,b), (c,c), (d,d), (a,b)\}.$$
   (a) Is R reflexive?
   (b) Is R symmetric?
   (c) Is R transitive?
   (d) Is R a function?

5. Negate the statement

Every subset of $\mathbb{Z}$ is well-ordered.

6. One can prove by induction that inequality $2n+1 \leq 2^n$ holds for all $n \geq 3$.
   (a) State the Induction Base
   (b) State the Induction Hypothesis.

7. Give counter examples to prove that the following statements are false:
   (a) Let $x, y \in \mathbb{Z}$. If $x < y$ then $x^2 < y^2$.
   (b) The relation R on $\mathbb{Q}$ defined by $xRy \iff x^2 + y^2 = 1$ is an equivalence relation.

8. Mark with a ✓ each statement that is proved by the following argument.

   If n is not a prime, then n can be factored as $n = pq$. Now one has

   $$2^n - 1 = 2^{pq} - 1 = (2^p)^q - 1 = (2^p - 1)((2^p)^{q-1} + (2^p)^{q-2} + \cdots + 2^p + 1)$$

   so $2^n - 1$ factors as well.

   (a) Let $n \in \mathbb{N}$. If n is a prime, then $2^n - 1$ is a prime.
   (b) Let $n \in \mathbb{N}$. If $2^n - 1$ is a prime, then n is a prime.
   (c) A natural number n is a prime if and only if $2^n - 1$ is a prime.
   (d) None of the above.

9. Give an example to show that the claim below is false and identify the flaw in the "proof".

   Claim: For every $a > 0$ one has $\sqrt{1+3a} \geq 1 + \sqrt{a}$.

   Proof: One has

   $$1 + 3a = 1 + 2a + a \geq 1 + 2\sqrt{a} + a = (1 + \sqrt{a})^2$$

   and, therefore, $\sqrt{a+3a} \geq 1 + \sqrt{a}$.

10. Conjecture a formula for a_n for $n \geq 1$ given

    $a_1 = 1, \quad a_2 = 2, \quad a_3 = 3, \quad a_4 = 5, \quad a_5 = 8$.

Part II

Solve five (5) of the problems 11–16; they count for 12% each. Your solutions to these problems must be written in a blue book. Turn in your solutions to five problems only!

Books, notes, and old homework are allowed aids on this part of the test. Calculators are also allowed, but phones, PDAs, Apple watches etc. are not.

11. Let A, B, and C be sets and consider functions $f\colon A \to B$ and $g, h\colon B \to C$. Show that if f is onto and $g \circ f = h \circ f$ then one has $g = h$.

12. Prove or disprove that the relation R on $\mathbb{R} \times \mathbb{R}$ defined by

$$(x, y)R(u, v) \iff |x| + |y| = |u| + |v|$$

is an equivalence relation.

13. Consider the statement

$$\forall x \in \mathbb{Z}, x \textit{ is odd} \implies 4|(x^2 - 5).$$

Give a

(a) direct proof,

(b) proof by contrapositive,

(c) proof by contradiction.

14. Let R_1 and R_2 be equivalence relations on a set A.

(a) Show that $R_1 \cap R_2$ is an equivalence relation on A.

(b) Give an example of a set A and equivalence relations R_1 and R_2 such that $R_1 \cup R_2$ is not an equivalence relation on A.

15. For sets A and B the set $A \bigtriangleup B = (A - B) \cup (B - A)$ is called the *symmetric difference.*

(a) Prove the equality $A \bigtriangleup B = A \cup B - A \cap B$.

(b) Prove that

$$X \cap (A \bigtriangleup B) = (X \cap A) \bigtriangleup (X \cap B)$$

holds for all sets A, B, and X.

16. Prove that

$$\sum_{u=0}^{n} u^2 = 2^{n+1} - 1$$

holds for all $n \geq 0$.